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Linear mixed-effects model for longitudinal complex data with diversified characteristics

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Conflicts of interest

The authors declare no conflict of interest.

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Abstract

The increasing richness of data encourages a comprehensive understanding of economic and financial activities, where variables of interest may include not only scalar (point-like) indicators, but also functional (curve-like) and compositional (pie-like) ones. In many research topics, variables are also chronologically collected across individuals, which falls into the paradigm of longitudinal analysis. The complicated nature of data, however, increases the difficulty of modeling these variables under a traditional longitudinal framework. In this study, we investigate a linear mixed-effects model (LMM) for such complex data. Different types of variables are first consistently represented using the corresponding basis expansions so that the LMM can then be conducted on them, which generalizes the theoretical framework of the LMM to complex data analysis. A number of numerical experiments indicate the feasibility and effectiveness of the proposed model. We further illustrate its practical utility in a real data study of China's stock market and show that the proposed method can enhance the performance and interpretability of the regression for complex data with diversified characteristics.

Key Words: Longitudinal complex data; Linear mixed-effects model; Compositional data analysis; Functional data analysis; Stock market; Online investors' emotions

1 Introduction

The development of sensors, information storage, and data mining makes it possible to collect data from a large number of sources with different characteristics such as familiar single points, curves, and pie charts. Multiple types of data, referred to as *complex data*, enlarge the traditional category of variables and provide researchers with an opportunity to understand the behavior of activities more comprehensively than ever before. For example, public online emotion from social media and intraday stock returns are improving the accuracy and interpretation of trend predictions in China's stock market (Wang et al., 2019a). In such a case, the return series are processed as continuous functions from opening to closing and investors' emotions are measured as compositions constituted by five types of emotions.

Complex data analysis involves various types of data ranging from classical nominal, ordinal, and ratio scalar variables to curve- and pie-like functional and compositional variables and even text data.

30 In this study, we focus on two emerging types, namely functions and compositions. Specifically, in
31 functional data analysis (FDA), a data unit is assumed to be a square-integrable function determined
32 by its observations at various times (Ramsay, 1982). The internal property of functions (i.e., the
33 infinite dimension) causes great difficulty for functional modeling in both theory and practice. To
34 describe functions over a bounded closed set (e.g., an interval), some equivalent representations such
35 as basis function expansion and the reproducing kernel method are thus necessary (Härdle et al., 2012).
36 On the contrary, compositional data analysis (CDA) discusses the intrinsic structure of a whole, such
37 as the proportions or percentages that carry only relative information (Aitchison, 1982, 1986). The
38 defining features of compositions include the strict positive and constant sum of all the components
39 inside (e.g., 1 for proportions and 100 for percentages), which is also problematic for most traditional
40 statistical approaches. To eliminate these strong constraints, a family of logratio transformations has
41 been proposed such as additive logratio, centered logratio (Aitchison, 1986), and isometric logratio
42 (Egozcue et al., 2003), abbreviated to the well-known *alr*, *clr*, and *ilr* transformations, respectively.
43 For further details on FDA and CDA, see Ramsay and Silverman (2005) and Pawlowsky-Glahn et al.
44 (2015), respectively.

45 Numerous works have investigated regression for functional and compositional covariates against
46 a scalar response. Ramsay and Silverman (2005, 2007) systematically proposed the theoretical frame-
47 work of functional linear regression and Müller and Stadtmüller (2005) then expanded it to the gen-
48 eralized linear case. More recent studies of functional regression follow the additive model (Fan et al.,
49 2015), mixture of linear models (Wang et al., 2016), and truncated linear model (Hall and Hooker,
50 2016). Meanwhile, Aitchison and Bacon-Shone (1984) initially proposed linear regression for composi-
51 tional covariates, Marzio et al. (2015) presented the kernel-based compositional regression, and Bruno
52 et al. (2014, 2016) investigated the spatiotemporal model and another nonparametric regression with
53 Bayesian P-splines, respectively.

54 The aforementioned approaches focus on a specific type of complex data, with relatively few
55 studies examining the multi-type situation. Wang et al. (2015) preliminarily developed a linear model
56 for multiple types of complex data. Wang et al. (2019a) then extended its computational framework to
57 generalized linear regression including scalar, functional, and compositional covariates. Their methods
58 were performed under the independent and identically distributed (IID) assumption of errors, namely
59 that all the samples of complex data are assumed to be independently collected from an identical
60 population. However, this IID assumption is problematic in some cases; these complex data may
61 also show typical longitudinal features, which we call *longitudinal complex data*. For example, as
62 introduced in Section 5, the closing prices of China's stock market were collected from hundreds of
63 stocks over several months. The price-limiting mechanism of the market results in high correlation

among observations of the same stock. In such a case, statistical models of complex data based on the IID assumption can be biased and lead to confusing results since such longitudinal features are ignored. Similar problems have been discussed separately for FDA (Goldsmith et al., 2012; Gertheiss et al., 2013; Chen and Cao, 2017) and CDA (Zhang et al., 2009; Qiu et al., 2010; Wang et al., 2019b), and few of the proposed solutions show the potential to integrate multiple types of complex data. Thus, developing a unified framework to model longitudinal complex data with diversified characteristics is necessary.

As a fundamental longitudinal technique, the linear mixed-effects model (LMM) proposed by Laird and Ware (1982) has been extended to numerous applications (Fitzmaurice et al., 2011; Hsiao, 2014), but most studies focus only on scalar variables. In this study, we investigate the LMM for complex data with diversified characteristics (CompLMM hereafter) to deal with longitudinal features. Specifically, we assume that the data are collected from N individuals along with n_i measurements for the i -th individual ($i = 1, 2, \dots, N$); then, CompLMM is formulated as

$$y_{ij} = \sum_{k=1}^{p_x} x_{ijk} \alpha_k + \sum_{k=1}^{q_z} z_{ijk} a_{ik} + \sum_{k=1}^{p_\mu} \int \mu_{ijk} \beta_k + \sum_{k=1}^{q_\nu} \int \nu_{ijk} b_{ik} + \sum_{k=1}^{p_c} (\mathbf{c}_{ijk}, \boldsymbol{\gamma}_k)_a + \sum_{k=1}^{q_w} (\mathbf{w}_{ijk}, \mathbf{r}_{ik})_a + \varepsilon_{ij} \quad (1)$$

for the j -th sample ($j = 1, 2, \dots, n_i$). Here, y_{ij} denotes the scalar response; x_{ijk} (z_{ijk}), μ_{ijk} (ν_{ijk}), and \mathbf{c}_{ijk} (\mathbf{w}_{ijk}) are constituted by part of the scalar, functional, and compositional covariates, with numbers of p_x (q_z), p_μ (q_ν), and p_c (q_w), respectively; α_k (a_{ik}), β_k (b_{ik}), and $\boldsymbol{\gamma}_k$ (\mathbf{r}_{ik}) denote the regression coefficients with the corresponding characteristics; ε_{ij} is the random scalar error; and $(\cdot, \cdot)_a$ denotes the Aitchison inner product in CDA to be introduced in Section 2. In the paradigm of the LMM, the terms containing α_k , β_k , and $\boldsymbol{\gamma}_k$ in Model (1) comprise the fixed effects shared by all individuals, whereas those containing a_{ik} , ν_{ik} , and \mathbf{r}_{ik} comprise the random effects unique to the specific one.

In particular, when there are no random effects, Model (1) is categorized as the computational framework of complex data in Wang et al. (2019a) and this reduces to the IID-based linear model (Wang et al., 2015), say CompLM, as

$$y_{ij} = \sum_{k=1}^{p_x} x_{ijk} \alpha_k + \sum_{k=1}^{p_\mu} \int \mu_{ijk} \beta_k + \sum_{k=1}^{p_c} (\mathbf{c}_{ijk}, \boldsymbol{\gamma}_k)_a + \varepsilon_{ij}.$$

Compared with CompLM, the introduction of random effects into Model (1) makes it possible to capture the subject-specific information of each individual on the basis of the common regression characteristics in the population. It also distinguishes the between- and within-subject variability of responses, which further improves the performance of linear regression on longitudinal complex data.

In this study, we estimate the parameters for Model (1). To consistently represent data with diversified characteristics, we first transform the functions and compositions inside using related basis

94 expansions such that they are described equivalently to numeric coordinates. These processed data
 95 are available to conduct the LMM and obtain the intermediate result that can then be reconstructed
 96 to match the original diversified characteristics. **Then, the necessary theoretical properties for the**
 97 **proposed longitudinal framework are developed accordingly.** To further measure the variability of
 98 different types of variables across individuals, we adopt the point-wise variance function and total
 99 variance for a random function and composition, respectively. The proposed CompLMM improves the
 100 regression of complex data with diversified characteristics and enhances its interpretability, and may
 101 provide an instructive unified framework for modeling longitudinal complex data.

102 The remainder of this paper is organized as follows. In Section 2, we review some fundamental
 103 knowledge on FDA and CDA. In Section 3, we investigate CompLMM and propose its computational
 104 algorithm. A series of simulation studies are then conducted to assess the performance of the proposed
 105 method, with the results presented in Section 4. Section 5 describes a real data study on China's stock
 106 market to illustrate the effectiveness of the proposed method. **Finally, some discussions and prospects**
 107 **are given in Section 6.**

108 2 Preliminaries

109 We briefly introduce the basic ideas and mathematical techniques for FDA and CDA, including basis
 110 function expansion for functions and ilr transformation for compositions. These provide the theoretical
 111 and computational foundation for the proposed method. For simplification, we use commas and
 112 semicolons in the matrix expressions to indicate that the adjacent blocks in a matrix are organized by
 113 column and row, respectively.

114 2.1 FDA

115 In FDA, a series of discrete data are considered to be collected from a potential single entity (i.e., the
 116 function) over time. Basis function expansion is one of the most practical methods of describing the
 117 continuous characteristics of a function (Ramsay and Silverman, 2005). That is, a function is expressed
 118 as a linear combination of the given basis functions, which can be realized using ordinary least squares
 119 (OLS), penalized OLS, or regularized principal components (Hall and Horowitz, 2007). Without loss
 120 of generality, we adopt B-spline basis functions and perform the simple OLS-based expansion in this
 121 study.

122 Specifically, given a group of basis functions $\{\phi_j\}_{j=1}^{\infty}$ over an interval \mathcal{I} , any square-integrable
 123 function, say $\mu \in \mathcal{L}_2$, can be formulated as $\mu = \sum_j u_j \phi_j$ with an infinite series of expansion coefficients
 124 u_j . When n samples, say o_i at time $t_i \in \mathcal{I}$ ($i = 1, 2, \dots, n$), are observed from μ , they are assumed
 125 subject to $o_i = \mu(t_i) + \epsilon_i$ with white noise ϵ_i . Then, the expansion of μ leads to $\mathbf{o} = \mathbf{\Phi}\mathbf{u} + \boldsymbol{\epsilon}$, where

126 $\mathbf{o} = (o_1, o_2, \dots, o_n)'$ and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)'$ in \mathbb{R}^n , $\boldsymbol{\Phi} = (\boldsymbol{\phi}(t_1), \boldsymbol{\phi}(t_2), \dots, \boldsymbol{\phi}(t_n))' \in \mathbb{R}^{n \times K}$ with
 127 $\boldsymbol{\phi}(t_i) = (\phi_1(t_i), \phi_2(t_i), \dots, \phi_K(t_i))' \in \mathbb{R}^K$, and $\mathbf{u} = (u_1, u_2, \dots, u_K)' \in \mathbb{R}^K$. In practice, the number
 128 of basis functions and related expansion coefficients K are limited below n because of the finite size
 129 of observations. Thus, the OLS estimation results in the truncated expansion coefficients of \mathbf{u} :

$$\mathbf{u} = (\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'\mathbf{o}. \quad (2)$$

130 The expansion coefficients greatly concentrate the features of the original function. Typically, the
 131 image of μ can be described explicitly in a point-wise manner as

$$\mu(t) = \sum_{j=1}^K u_j \phi_j(t) = \mathbf{u}'\boldsymbol{\phi}(t) \quad (t \in \mathcal{I}), \quad (3)$$

132 where μ is determined completely by \mathbf{u} along with the known basis functions $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_K)'$.
 133 The expectation of μ is also associated with that of \mathbf{u} , and the variance function for μ , denoted by
 134 \mathcal{K}_μ , can be formulated as

$$\mathcal{K}_\mu(t) = \text{Var}(\mu(t)) = \boldsymbol{\phi}'(t)\text{Var}(\mathbf{u})\boldsymbol{\phi}(t) \quad (t \in \mathcal{I}), \quad (4)$$

135 where $\text{Var}(\cdot)$ denotes the covariance matrix of a random vector. Moreover, the integral of the product
 136 of two functions, say μ and β with its expansion coefficients $\boldsymbol{\lambda}$, can be written as

$$\int \mu(t)\beta(t)dt = \mathbf{u}'\mathbf{W}\boldsymbol{\lambda} \quad \text{with} \quad \mathbf{W} = \int \boldsymbol{\phi}(t)\boldsymbol{\phi}'(t)dt. \quad (5)$$

137 To compute the integral in \mathbf{W} numerically, we uniformly sample from \mathcal{I} , say $\{\tau_1, \tau_2, \dots, \tau_T\}$ with
 138 T points and approximate it as $\mathbf{W} = T^{-1} \sum_{i=1}^T \boldsymbol{\phi}(\tau_i)\boldsymbol{\phi}'(\tau_i)$. \mathbf{W} can also be adopted to express the
 139 overall difference between two functions as

$$d_{\mathcal{L}_2}^2(\beta, \hat{\beta}) = \int (\beta(t) - \hat{\beta}(t))^2 dt = (\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}})'\mathbf{W}(\boldsymbol{\lambda} - \hat{\boldsymbol{\lambda}}) \quad (6)$$

140 with $\hat{\beta} \in \mathcal{L}_2$ and its expansion coefficients $\hat{\boldsymbol{\lambda}}$. These properties of basis function expansion make it
 141 possible to equivalently represent infinite-dimensional functions as relatively few numeric variables.

142 2.2 CDA

143 In CDA, the attraction of a multivariate vector is the relative magnitude, instead of the absolute one,
 144 among all the components inside. Working with this scale invariance property, any composition with
 145 D inner parts, say \mathbf{c} , can be expressed as $\mathbf{c} = (c_1, c_2, \dots, c_D)'$ subject to $c_i > 0$ ($i = 1, 2, \dots, D$) and
 146 $\mathbf{c}'\mathbf{1}_D = 1$ with $\mathbf{1}_D$ constituted by 1 in \mathbb{R}^D . All such D -part compositions consist of the D -dimensional
 147 simplex space denoted by S^D .

148 To remove the constraints of compositions, Egozcue et al. (2003) proposed the ilr transformation
 149 via the simplicial orthonormal basis. In this study, we follow Egozcue and Pawłowsky-Głahn (2005)

150 and represent any composition as its specified coordinates. Take \mathbf{c} , for example, $\text{ilr}(\mathbf{c}) = \mathbf{c}^* =$
 151 $(c_1^*, c_2^*, \dots, c_{D-1}^*)'$, where

$$c_i^* = \frac{1}{\sqrt{(D-i+1)(D-i)}} \sum_{j=1}^{D-i} \log c_j - \sqrt{\frac{D-i}{D-i+1}} \log c_{D-i+1} \quad (i = 1, 2, \dots, D-1). \quad (7)$$

152 These coordinates contain all the relative information on \mathbf{c} ; therefore, they can be used to reconstruct
 153 the original composition. That is, $\mathbf{c} = \text{ilr}^{-1}(\mathbf{c}^*) = \mathcal{C}(\exp(\boldsymbol{\omega}))$, where $\mathcal{C}(\cdot)$ denotes the closure operation
 154 that scales a vector with positive components proportionally such that it conforms to the constraints
 155 of compositions, and $\exp(\boldsymbol{\omega}) = (\exp \omega_1, \exp \omega_2, \dots, \exp \omega_D)'$ with

$$\omega_i = \sum_{j=0}^{D-i} \frac{c_j^*}{\sqrt{(D-j+1)(D-j)}} - \sqrt{\frac{i-1}{i}} c_{D-i+1}^* \quad (i = 1, 2, \dots, D) \quad (8)$$

156 and $c_0^* = c_D^* = 0$. Using the contrast matrix, denoted by $\boldsymbol{\Psi} \in \mathbb{R}^{(D-1) \times D}$, the ilr transformation and
 157 its inverse can be respectively expressed as $\text{ilr}(\mathbf{c}) = \boldsymbol{\Psi} \log(\mathbf{c})$ and $\text{ilr}^{-1}(\mathbf{c}^*) = \mathcal{C}(\exp(\boldsymbol{\Psi}' \mathbf{c}^*))$, where
 158 $\log(\mathbf{c}) = (\log c_1, \log c_2, \dots, \log c_D)'$. Specifically, $\boldsymbol{\Psi}$ associated with (7) and (8) is constituted by the
 159 elements ψ_{ij} as $\psi_{ij} = \sqrt{\frac{D-i}{D-i+1}} \rho_{ij}$ for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, n_i$, where $\rho_{ij} = (D-i)^{-1}$ when
 160 $j < D-i+1$, $\rho_{ij} = -1$ when $j = D-i+1$, and $\rho_{ij} = 0$ otherwise.

161 As an isometry between the simplex and Euclidian spaces, the ilr transformation facilitates the
 162 computation of the Aitchison geometry. For example, the Aitchison inner product, denoted by $(\cdot, \cdot)_a$,
 163 can be easily expressed as

$$(\mathbf{c}, \boldsymbol{\gamma})_a = \text{ilr}(\mathbf{c})' \text{ilr}(\boldsymbol{\gamma}) \quad (\boldsymbol{\gamma} \in S^D). \quad (9)$$

164 The related norm and distance, denoted by $\|\cdot\|_a$ and $d_a(\cdot, \cdot)$, then follow respectively as

$$\|\boldsymbol{\gamma}\|_a^2 = (\boldsymbol{\gamma}^*)' \boldsymbol{\gamma}^* \quad \text{and} \quad d_a^2(\boldsymbol{\gamma}, \hat{\boldsymbol{\gamma}}) = (\boldsymbol{\gamma}^* - \hat{\boldsymbol{\gamma}}^*)' (\boldsymbol{\gamma}^* - \hat{\boldsymbol{\gamma}}^*)$$

165 with $\hat{\boldsymbol{\gamma}} \in S^D$ and its ilr coordinates $\hat{\boldsymbol{\gamma}}^*$. Moreover, the total variance of \mathbf{c} , denoted by $\text{totVar}(\mathbf{c})$, can
 166 be decomposed as

$$\text{totVar}(\mathbf{c}) = \sum_{i=1}^{D-1} \text{Var}(c_i^*) \quad (10)$$

167 using the ilr coordinates. Since $\boldsymbol{\Psi}' \boldsymbol{\Psi}$ is identically equal to $\mathbf{I}_D - \mathbf{1}_D \mathbf{1}'_D / D$, where \mathbf{I}_D denotes the
 168 D -dimensional unit matrix (Pawlowsky-Glahn et al., 2015), the specified ilr transformation here does
 169 not affect those results above. From these properties of the ilr transformation, we can substitute the
 170 ilr coordinates with no constraints for the compositional covariates in most statistical models.

171 3 LMM for complex data

172 In this section, we investigate the LMM for longitudinal complex data with diversified characteristics.
 173 The approach used to aggregate multiple types of complex data along with their properties and some
 174 issues in practice are also discussed.

175 3.1 Model

176 To uniformly represent complex data with different characteristics, we apply the B-spline expansion
177 and ilr transformation to Model (1). Thus, the model can be formulated using (5) and (9) as

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\alpha} + \mathbf{z}'_{ij}\mathbf{a}_i + \sum_{k=1}^{p_\mu} \mathbf{u}'_{ijk}\mathbf{W}\boldsymbol{\lambda}_k + \sum_{k=1}^{q_\nu} \mathbf{v}'_{ijk}\mathbf{W}\boldsymbol{\theta}_{ik} + \sum_{k=1}^{p_c} (\mathbf{c}^*_{ijk})'\boldsymbol{\gamma}_k^* + \sum_{k=1}^{q_w} (\mathbf{w}^*_{ijk})'\mathbf{r}_{ik}^* + \varepsilon_{ij},$$

178 where $\mathbf{x}_{ij} = (x_{ij1}, x_{ij2}, \dots, x_{ij,p_x})' \in \mathbb{R}^{p_x}$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{p_x})'$ and $\mathbf{z}_{ij} = (z_{ij1}, z_{ij2}, \dots, z_{ij,q_z})' \in$
179 \mathbb{R}^{q_z} with $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{i,q_z})'$; \mathbf{u}_{ijk} , \mathbf{v}_{ijk} , $\boldsymbol{\lambda}_k$ and $\boldsymbol{\theta}_{ik}$ denote the expansion coefficients of μ_{ijk} , ν_{ijk} ,
180 β_k and b_{ik} , respectively, with a common dimension K ; and \mathbf{c}^*_{ijk} , $\boldsymbol{\gamma}_k^*$, \mathbf{w}^*_{ijk} and \mathbf{r}_{ik}^* denote the ilr
181 coordinates of the related compositions. To simplify, we further reformulate it as

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\alpha} + \mathbf{z}'_{ij}\mathbf{a}_i + \mathbf{u}'_{ij}\mathbf{W}_{p_\mu}\boldsymbol{\lambda} + \mathbf{v}'_{ij}\mathbf{W}_{q_\nu}\boldsymbol{\theta}_i + (\mathbf{c}^*_{ij})'\boldsymbol{\gamma}^* + (\mathbf{w}^*_{ij})'\mathbf{r}_i^* + \varepsilon_{ij}, \quad (11)$$

182 where $\mathbf{u}_{ij} = (\mathbf{u}_{ij1}; \mathbf{u}_{ij2}; \dots; \mathbf{u}_{ij,p_\mu}) \in \mathbb{R}^{Kp_\mu}$ and $\mathbf{v}_{ij} = (\mathbf{v}_{ij1}; \mathbf{v}_{ij2}; \dots; \mathbf{v}_{ij,q_\nu}) \in \mathbb{R}^{Kq_\nu}$, along with $\boldsymbol{\lambda} =$
183 $(\boldsymbol{\lambda}_1; \boldsymbol{\lambda}_2; \dots; \boldsymbol{\lambda}_{p_\mu})$ and $\boldsymbol{\theta}_i = (\boldsymbol{\theta}_{i1}; \boldsymbol{\theta}_{i2}; \dots; \boldsymbol{\theta}_{i,q_\mu})$, respectively; \mathbf{W}_{p_μ} (\mathbf{W}_{q_ν}) denotes the blocked diagonal
184 matrix consisting of p_μ (q_ν) matrices \mathbf{W} ; and $\mathbf{c}^*_{ij} = (\mathbf{c}^*_{ij1}; \mathbf{c}^*_{ij2}; \dots; \mathbf{c}^*_{ij,p_c}) \in \mathbb{R}^{(D-1)p_c}$ and $\mathbf{w}^*_{ij} =$
185 $(\mathbf{w}^*_{ij1}; \mathbf{w}^*_{ij2}; \dots; \mathbf{w}^*_{ij,q_w}) \in \mathbb{R}^{(D-1)q_w}$, along with $\boldsymbol{\gamma}^* = (\boldsymbol{\gamma}_1^*; \boldsymbol{\gamma}_2^*; \dots; \boldsymbol{\gamma}_{p_c}^*)$ and $\mathbf{r}_i^* = (\mathbf{r}_{i1}^*; \mathbf{r}_{i2}^*; \dots; \mathbf{r}_{i,q_w}^*)$,
186 respectively.

187 Jointly considering all the samples from the same individual, say the i -th one, we pile up n_i samples
188 from it by row. Finally, Model (11) can be rewritten as

$$\mathbf{y}_i = \mathbf{x}_i\boldsymbol{\alpha} + \mathbf{u}_i\mathbf{W}_{p_\mu}\boldsymbol{\lambda} + \mathbf{c}_i^*\boldsymbol{\gamma}^* + \mathbf{z}_i\mathbf{a}_i + \mathbf{v}_i\mathbf{W}_{q_\nu}\boldsymbol{\theta}_i + \mathbf{w}_i^*\mathbf{r}_i^* + \boldsymbol{\varepsilon}_i \quad (i = 1, 2, \dots, N), \quad (12)$$

189 where $\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{i,n_i})'$ and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2}, \dots, \varepsilon_{i,n_i})'$ in \mathbb{R}^{n_i} ; specifically, the fixed effects involve
190 $\mathbf{x}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \dots, \mathbf{x}_{i,n_i})' \in \mathbb{R}^{n_i \times p_x}$, $\mathbf{u}_i = (\mathbf{u}_{i1}, \mathbf{u}_{i2}, \dots, \mathbf{u}_{i,n_i})' \in \mathbb{R}^{n_i \times p_\nu}$ and $\mathbf{c}_i^* = (\mathbf{c}_{i1}^*, \mathbf{c}_{i2}^*, \dots, \mathbf{c}_{i,n_i}^*)' \in$
191 $\mathbb{R}^{n_i \times p_c}$, and the random effects involve $\mathbf{z}_i = (\mathbf{z}_{i1}, \mathbf{z}_{i2}, \dots, \mathbf{z}_{i,n_i})' \in \mathbb{R}_{n_i \times q_z}$, $\mathbf{v}_i = (\mathbf{v}_{i1}, \mathbf{v}_{i2}, \dots, \mathbf{v}_{i,n_i})' \in$
192 $\mathbb{R}^{n_i \times q_\nu}$ and $\mathbf{w}_i^* = (\mathbf{w}_{i1}^*, \mathbf{w}_{i2}^*, \dots, \mathbf{w}_{i,n_i}^*)' \in \mathbb{R}^{n_i \times q_w}$. To coincide with the paradigm of the LMM for
193 the scalar variables in Model (12), the total coefficients for the fixed and random effects refer to
194 $\boldsymbol{\varpi} = (\boldsymbol{\alpha}; \boldsymbol{\lambda}; \boldsymbol{\gamma}^*)$ and $\boldsymbol{\pi}_i = (\mathbf{a}_i; \boldsymbol{\theta}_i; \mathbf{r}_i^*)$, with the dimensions of $p = p_x + Kp_\mu + (D-1)p_c$ and $q =$
195 $q_z + Kq_\nu + (D-1)q_w$, respectively. Here, $\boldsymbol{\varpi}$ contains the common characteristics shared by the entire
196 population and $\boldsymbol{\pi}_i$ shows the specific ones of the i -th individual. Moreover, $\boldsymbol{\varepsilon}_i$ is assumed to obey the
197 normal distribution in \mathbb{R}^{n_i} , namely $\boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}_{n_i}, \sigma^2 \mathbf{I}_{n_i})$, where $\mathbf{0}_{n_i}$ is constituted by 0 in \mathbb{R}^{n_i} , and $\boldsymbol{\pi}_i$
198 is assumed to be independent of $\boldsymbol{\varepsilon}_i$ and normally distributed in \mathbb{R}^q , namely $\boldsymbol{\pi}_i \sim \mathcal{N}(\mathbf{0}_q, \mathbf{G})$, where
199 \mathbf{G} is positively defined and constant for all the individuals. Thus, the parameters to be estimated in
200 Model (12) include $\boldsymbol{\Theta} = \{\boldsymbol{\varpi}, \mathbf{G}, \sigma^2\}$.

201 Under these aforementioned assumptions, the estimate of $\boldsymbol{\Theta}$, denoted by $\widehat{\boldsymbol{\Theta}} = \{\widehat{\boldsymbol{\varpi}}, \widehat{\mathbf{G}}, \widehat{\sigma}^2\}$, can be
202 derived using the expectation maximum (EM) algorithm. Then, the fitted response, for example, \hat{y}_{ij}

203 in Model (11), can be expressed as

$$\hat{y}_{ij} = \mathbf{x}'_{ij}\hat{\boldsymbol{\alpha}} + \mathbf{z}'_{ij}\hat{\mathbf{a}}_i + \mathbf{u}'_{ij}\mathbf{W}_{p\mu}\hat{\boldsymbol{\lambda}} + \mathbf{v}'_{ij}\mathbf{W}_{qv}\hat{\boldsymbol{\theta}}_i + (\mathbf{c}_{ij}^*)'\hat{\boldsymbol{\gamma}}^* + (\mathbf{w}_{ij}^*)'\hat{\mathbf{r}}_i^*,$$

204 or $\hat{y}_{ij} = \mathbf{x}'_{ij}\hat{\boldsymbol{\alpha}} + \mathbf{u}'_{ij}\mathbf{W}_{p\mu}\hat{\boldsymbol{\lambda}} + (\mathbf{c}_{ij}^*)'\hat{\boldsymbol{\gamma}}^*$ for the reduced CompLM, where $\widehat{\boldsymbol{\varpi}}$ consists of $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{\gamma}}^*$, and
 205 $\hat{\mathbf{a}}_i$, $\hat{\boldsymbol{\theta}}_i$ and $\hat{\mathbf{r}}_i^*$ can be obtained from $\hat{\boldsymbol{\Theta}}$. Before we develop the estimation procedure, we discuss the
 206 relationship between the original and reconstructed models (i.e., Models (1) and (12)) in the following
 207 remarks.

208 **Remark 1.** The assumption of the independence of the random effects coefficients and errors is
 209 important for the theory of the LMM. We put this assumption on the reconstructed unified numeric
 210 variables, namely $\boldsymbol{\theta}_i$ and \mathbf{r}_i^* , in Model (12), which implies the independence of the original coefficients
 211 with diversified characteristics (e.g., b_{ik} and \mathbf{r}_{ik}) and the scalar errors in Model (1). The covariance
 212 between b_{ik} with functional characteristics and ε_{ij} is defined as point-wise (Ramsay and Silverman,
 213 2005), namely $\text{Cov}(b_{ik}(t), \varepsilon_{ij})$ for any $t \in \mathcal{I}$. Under the given basis functions $\boldsymbol{\phi}$, the expectations of
 214 the related expansion coefficients are equal to zero; then, we have

$$\text{Cov}(b_{ik}(t), \varepsilon_{ij}) = \boldsymbol{\theta}'_{ik}\mathbb{E}[\boldsymbol{\phi}(t)\varepsilon_{ij}] = \boldsymbol{\phi}'(t)\text{Cov}(\boldsymbol{\theta}_{ik}, \varepsilon_{ij}). \quad (13)$$

215 From (13), the independence assumption on the expansion coefficients, namely $\text{Cov}(\boldsymbol{\theta}_{ik}, \varepsilon_{ij}) = \mathbf{0}_K$, is
 216 sufficient to that on the original overall function. On the contrary, the covariance between \mathbf{r}_{ik} with the
 217 compositional characteristics and ε_{ij} is directly defined using the ilr coordinates, namely $\text{Cov}(\mathbf{r}_{ik}^*, \varepsilon_{ij})$,
 218 and the independence assumption holds for any specified ilr transformation (Wang et al., 2019b).

219 **Remark 2.** Another issue for the theory of the LMM follows the covariance matrix of the random
 220 effects coefficients. For functional variables, this refers to the covariance function, say $\mathcal{K}_{b_{ik}, b_{ik'}}(s, t)$ for
 221 b_{ik} and $b_{ik'}$ with the expansion coefficients $\boldsymbol{\theta}_{ik'}$ at times s and t . Similar to (4), it can be formulated
 222 as

$$\mathcal{K}_{b_{ik}, b_{ik'}}(s, t) = \text{Cov}(b_{ik}(s), b_{ik'}(t)) = \boldsymbol{\phi}'(s)\text{Cov}(\boldsymbol{\theta}_{ik}, \boldsymbol{\theta}_{ik'})\boldsymbol{\phi}(t).$$

223 Specifically, it reduces to $\mathcal{K}_{b_{ik}}(s, t) = \boldsymbol{\phi}'(s)\text{Var}(\boldsymbol{\theta}_{ik})\boldsymbol{\phi}(t)$ when $k' = k$. For compositional variables,
 224 say \mathbf{r}_{ik} and $\mathbf{r}_{ik'}$ with the ilr coordinates $\mathbf{r}_{ik'}^*$, the covariance matrix, as Mateu-Figueras et al. (2013)
 225 suggested, can be naturally defined as $\text{Cov}(\mathbf{r}_{ik}, \mathbf{r}_{ik'}) = \text{Cov}(\mathbf{r}_{ik}^*, \mathbf{r}_{ik'}^*)$. Then, the covariance matrix for
 226 the two types of variables can be defined consistently. For example, we express the covariance function
 227 for b_{ik} and \mathbf{r}_{ik} at time t , denoted by $\mathcal{K}_{b_{ik}, \mathbf{r}_{ik}}(t)$, as

$$\mathcal{K}_{b_{ik}, \mathbf{r}_{ik}}(t) = \text{Cov}(b_{ik}(t), \mathbf{r}_{ik}^*) = \boldsymbol{\phi}'(t)\text{Cov}(\boldsymbol{\theta}_{ik}, \mathbf{r}_{ik}^*).$$

228 In those cases, the different patterns inside the covariance matrix for the original model are described
 229 by the elements of \mathbf{G} , including $\text{Cov}(\boldsymbol{\theta}_{ik}, \boldsymbol{\theta}_{ik'})$, $\text{Var}(\boldsymbol{\theta}_{ik})$, $\text{Cov}(\mathbf{r}_{ik}^*, \mathbf{r}_{ik'}^*)$, and $\text{Cov}(\boldsymbol{\theta}_{ik}, \mathbf{r}_{ik}^*)$. Thus, \mathbf{G} in
 230 the reconstructed model concentrates the covariance structure of multiple types of complex data.

231 3.2 Parameter estimation

232 When there are no random effects in Model (12), as considered in ComplLM (Wang et al., 2015), the
 233 OLS-based estimates of $\boldsymbol{\varpi}$ and σ^2 , denoted by $\widehat{\boldsymbol{\varpi}}_{ols}$ and $\hat{\sigma}_{ols}^2$, have explicit solutions, that is,

$$\widehat{\boldsymbol{\varpi}}_{ols} = \left(\sum_{i=1}^N \mathbb{X}_i' \mathbb{X}_i \right)^{-1} \left(\sum_{i=1}^N \mathbb{X}_i' \mathbf{y}_i \right), \quad (14)$$

$$\hat{\sigma}_{ols}^2 = M^{-1} \sum_{i=1}^N (\mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\varpi}}_{ols})' (\mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\varpi}}_{ols}), \quad (15)$$

234 where $\mathbb{X}_i = (\mathbf{x}_i, \mathbf{u}_i \mathbf{W}_\mu, \mathbf{c}_i^*) \in \mathbb{R}^{n_i \times p}$ for $i = 1, 2, \dots, N$ and $M = \sum_{i=1}^N n_i$. Here, $\widehat{\boldsymbol{\varpi}}_{ols}$ and $\hat{\sigma}_{ols}^2$ also
 235 indicate the consistent estimates within the computational framework of Wang et al. (2019a) for linear
 236 regression since both studies imply the same model expression as ComplLM.

237 In general, the estimation procedure for Model (12) can be implemented using the EM algorithm
 238 (Laird et al., 1987). Specifically, given a pair of estimates $\widehat{\mathbf{G}}^{(\omega)}$ and $(\hat{\sigma}^{(\omega)})^2$, where the superscript ω
 239 indicates the iteration and $\omega = 0$ denotes the initial values, $\widehat{\boldsymbol{\varpi}}^{(\omega)}$ is formulated as

$$\widehat{\boldsymbol{\varpi}}^{(\omega)} = \left(\sum_{i=1}^N \mathbb{X}_i' (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1} \mathbb{X}_i \right)^{-1} \left(\sum_{i=1}^N \mathbb{X}_i' (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1} \mathbf{y}_i \right) \quad (16)$$

240 with

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)} = \mathbb{Z}_i \widehat{\mathbf{G}}^{(\omega)} \mathbb{Z}_i' + (\hat{\sigma}^{(\omega)})^2 \mathbf{I}_{n_i} \quad (17)$$

241 and $\mathbb{Z}_i = (\mathbf{z}_i, \mathbf{v}_i \mathbf{W}_{q\nu}, \mathbf{w}_i^*) \in \mathbb{R}^{n_i \times q}$ for $i = 1, 2, \dots, N$. On the contrary, when $\widehat{\boldsymbol{\varpi}}^{(\omega)}$ is available, $\widehat{\mathbf{G}}^{(\omega)}$
 242 and the others can be updated from $(\hat{\sigma}^{(\omega)})^2$, that is,

$$\widehat{\mathbf{G}}^{(\omega+1)} = N^{-1} \sum_{i=1}^N (\widehat{\boldsymbol{\pi}}_i^{(\omega)} (\widehat{\boldsymbol{\pi}}_i^{(\omega)})' + \widehat{\mathbf{G}}^{(\omega)} - \widehat{\mathbf{G}}^{(\omega)} \mathbb{Z}_i' (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1} \mathbb{Z}_i \widehat{\mathbf{G}}^{(\omega)}), \quad (18)$$

$$(\hat{\sigma}^{(\omega+1)})^2 = M^{-1} \sum_{i=1}^N ((\widehat{\mathbf{e}}_i^{(\omega)})' \widehat{\mathbf{e}}_i^{(\omega)} + (\hat{\sigma}^{(\omega)})^2 \text{tr}(\mathbf{I}_{n_i} - (\hat{\sigma}^{(\omega)})^2 (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1})), \quad (19)$$

243 where

$$\widehat{\boldsymbol{\pi}}_i^{(\omega)} = \widehat{\mathbf{G}}^{(\omega)} \mathbb{Z}_i' (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1} (\mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\varpi}}^{(\omega)}), \quad (20)$$

$$\widehat{\mathbf{e}}_i^{(\omega)} = \mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\varpi}}^{(\omega)} - \mathbb{Z}_i \widehat{\boldsymbol{\pi}}_i^{(\omega)} \quad (21)$$

244 and $\text{tr}(\cdot)$ denotes the trace of a matrix. Such $\widehat{\mathbf{G}}^{(\omega+1)}$ and $(\hat{\sigma}^{(\omega+1)})^2$ result in the update $\widehat{\boldsymbol{\varpi}}^{(\omega+1)}$ from
 245 (16), which finishes an iteration.

246 Under the framework of the EM algorithm, the proposed procedure is always convergent since the
 247 quadratic convex optimization is involved. The convergence criterion is that the maximum difference
 248 between the present estimated parameters, say $\widehat{\boldsymbol{\varpi}}^{(\omega)}$ and $(\hat{\sigma}^{(\omega)})^2$, and the previous ones, say $\widehat{\boldsymbol{\varpi}}^{(\omega-1)}$
 249 and $(\hat{\sigma}^{(\omega-1)})^2$, falls into a given threshold, say $\delta = 0.01$, that is,

$$\max \left\{ \left\| \widehat{\boldsymbol{\varpi}}^{(\omega)} - \widehat{\boldsymbol{\varpi}}^{(\omega-1)} \right\|_\infty, \left| (\hat{\sigma}^{(\omega)})^2 - (\hat{\sigma}^{(\omega-1)})^2 \right| \right\} < \delta, \quad (22)$$

250 where $\|\cdot\|_\infty$ denotes the maximum norm of a vector. Meanwhile, the algorithm also stops if it exceeds
 251 the iteration limit, say $l = 100$. As suggested by Laird et al. (1987), the initial value of $\widehat{\boldsymbol{\omega}}$, denoted
 252 by $\widehat{\boldsymbol{\omega}}^{(0)}$, is set to be $\widehat{\boldsymbol{\omega}}_{ols}$, and those of the other parameters can then be computed from $\widehat{\boldsymbol{\omega}}^{(0)}$ as

$$\widehat{\mathbf{G}}^{(0)} = N^{-1} \sum_{i=1}^N (\widehat{\boldsymbol{\pi}}_i^{(0)} (\widehat{\boldsymbol{\pi}}_i^{(0)})' - (\widehat{\sigma}^{(0)})^2 (\mathbb{Z}_i' \mathbb{Z}_i)^{-1}), \quad (23)$$

$$(\widehat{\sigma}^{(0)})^2 = L^{-1} \sum_{i=1}^N (\mathbf{y}_i - \mathbb{Z}_i \widehat{\boldsymbol{\pi}}_i^{(0)})' (\mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\omega}}^{(0)}), \quad (24)$$

253 where $L = M - (N - 1)q - p$ and $\widehat{\boldsymbol{\pi}}^{(0)} = (\mathbb{Z}_i' \mathbb{Z}_i)^{-1} \mathbb{Z}_i' (\mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\omega}}^{(0)})$. The aforementioned initialization
 254 for the estimation procedure begins with the reduced OLS-based linear regression (i.e., **CompLM**) and
 255 further abstracts the subject-specific information from the covariance structure of errors. Again, it
 256 verifies that the proposed **CompLMM** improves the performance of the final regression for longitudinal
 257 complex data compared with **CompLM**.

258 **Remark 3.** The proposed parameter estimation for **CompLMM** using the EM algorithm is consistent
 259 with the existing solutions for **CompLM** in (14) and (15) proposed by Wang et al. (2015, 2019a).
 260 Actually, when there are no random effects, namely no z_{ijk} , ν_{ijk} , and \mathbf{w}_{ijk} in Model (1), $\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)}$ reduces
 261 to become proportional to \mathbf{I}_{n_i} with no \mathbb{Z}_i involved in (17), implying that $\widehat{\boldsymbol{\omega}}^{(\omega)}$ in (16) equals $\widehat{\boldsymbol{\omega}}_{ols}$
 262 in (14) for any ω . A similar conclusion on (19) can also be drawn, namely that $(\widehat{\sigma}^{(\omega+1)})^2 \equiv \widehat{\sigma}_{ols}^2$ since

$$\widehat{\mathbf{e}}_i^{(\omega)} = \mathbf{y}_i - \mathbb{X}_i \widehat{\boldsymbol{\omega}}^{(\omega)} \quad \text{and} \quad (\widehat{\sigma}^{(\omega)})^2 (\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega)})^{-1} = \mathbf{I}_{n_i}$$

263 here. We conclude from these results that **CompLM** works exactly as the pooled method for longitu-
 264 dinal complex data.

265 In summary, Algorithm 1 presents the computational procedure of the proposed method for lon-
 266 gitudinal complex data.

Algorithm 1 Computational procedure for CompLMM

Input: The data set $\{(y_{ij}, x_{ijk}, z_{ijk}, o_{\mu_{ijk}}^m, o_{\nu_{ijk}}^m, \mathbf{c}_{ijk}, \mathbf{w}_{ijk}; t_{\mu_{ijk}}^m, t_{\nu_{ijk}}^m)\}_{i,j,k,m=1}^{N,n_i,p_*,n}$, with p_* corresponding to the related dimension, including the responses $\{y_{ij}\}_{i,j=1}^{N,n_i}$, scalar covariates $\{(x_{ijk}, z_{ijk})\}_{i,j,k=1}^{N,n_i,p_x/q_z}$, observations from functional covariates $\{(o_{\mu_{ijk}}^m, o_{\nu_{ijk}}^m)\}_{i,j,k,m=1}^{N,n_i,p_\mu/p_\nu,n}$ at times $\{(t_{\mu_{ijk}}^m, t_{\nu_{ijk}}^m)\}_{i,j,k,m=1}^{N,n_i,p_\mu/q_\nu,n}$, and compositional covariates $\{(\mathbf{c}_{ijk}, \mathbf{w}_{ijk})\}_{i,j,k=1}^{N,n_i,p_c/q_w}$; the given K basis functions $\{\phi_i\}_{i=1}^K$; the initial value of the parameter $\widehat{\boldsymbol{\omega}}^{(0)}$, associated with the intermediate $\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(0)}$; the convergence threshold δ ; and the iteration limit l .

Output: $\widehat{\boldsymbol{\Theta}} = \{\widehat{\boldsymbol{\omega}}, \widehat{\mathbf{G}}, \hat{\sigma}^2\}$ and $\widehat{\boldsymbol{\pi}}_i$ for $i = 1, 2, \dots, N$.

- 1: Compute the expansion coefficients \mathbf{u}_{ijk} and \mathbf{v}_{ijk} ($i = 1, 2, \dots, N; j = 1, 2, \dots, n_i$):

$$\begin{aligned}\mathbf{u}_{ijk} &= (\boldsymbol{\Phi}'_{\mu_{ijk}} \boldsymbol{\Phi}_{\mu_{ijk}})^{-1} \boldsymbol{\Phi}'_{\mu_{ijk}} \mathbf{o}_{\mu_{ijk}} \quad (k = 1, 2, \dots, p_\mu), \\ \mathbf{v}_{ijk} &= (\boldsymbol{\Phi}'_{\nu_{ijk}} \boldsymbol{\Phi}_{\nu_{ijk}})^{-1} \boldsymbol{\Phi}'_{\nu_{ijk}} \mathbf{o}_{\nu_{ijk}} \quad (k = 1, 2, \dots, p_\nu),\end{aligned}$$

where the notations coincide with (2) and the subscript indicates the functional covariate;

- 2: Compute the ilr coordinates \mathbf{c}_{ijk}^* and \mathbf{w}_{ijk}^* from (7);
- 3: Construct the data matrices \mathbb{X}_i and \mathbb{Z}_i ($i = 1, 2, \dots, N$); set $\omega = 0$;
- 4: **repeat**
- 5: Compute the intermediates $\widehat{\boldsymbol{\pi}}_i^{(\omega+1)}$ and $\widehat{\mathbf{e}}_i^{(\omega+1)}$ ($i = 1, 2, \dots, N$) from (20) and (21), respectively;
- 6: Update $\widehat{\mathbf{G}}^{(\omega+1)}$ and $(\hat{\sigma}^{(\omega+1)})^2$ from (18) and (19), respectively;
- 7: Update the intermediate $\widehat{\boldsymbol{\Sigma}}_{\mathbf{y}_i}^{(\omega+1)}$ ($i = 1, 2, \dots, N$) from (17);
- 8: Update $\widehat{\boldsymbol{\omega}}^{(\omega+1)}$ from (16);
- 9: Let $\omega := \omega + 1$;
- 10: **until** (22) holds or $\omega > l$;
- 11: **return**

$$\widehat{\boldsymbol{\Theta}} := \{\widehat{\boldsymbol{\omega}}^{(t)}, \widehat{\mathbf{G}}^{(t)}, (\hat{\sigma}^{(t)})^2\} \quad \text{and} \quad \widehat{\boldsymbol{\pi}}_i := \widehat{\boldsymbol{\pi}}_i^{(t)} \quad (i = 1, 2, \dots, N).$$

267 3.3 Some issues

268 In this study, we exemplify the proposed framework for longitudinal complex data using three types
 269 of covariates: scalar, function, and composition. This framework is actually available for more types
 270 of variables with diversified characteristics. For example, introducing dummy variables is common
 271 for processing categorical, nominal, and ordinal variables in longitudinal analysis (Hsiao, 2014). We
 272 can represent these as groups of dummy variables and conduct CompLMM for these multiple scalar
 273 covariates, where the order relation is ignored because of the continuous response. For unstructured

274 text data, we can summarize these into a series of compositions associated with the frequencies of
 275 topics or positions, similar to the measurement of investors' emotions by Zhou et al. (2017), and
 276 therefore analyze text data under the proposed framework.

277 The key technique for formulating CompLMM is to find a suitable representation of a specific type
 278 of complex data and related consistent algebraic system, such as the dummy variable expression for
 279 categorical, nominal, and ordinal covariates, basis function expansion with the l_2 norm for the Hilbert
 280 function space in FDA, and ilr transformation with the Aitchison inner product for the simplex in
 281 CDA. Following this idea, more diversified types of variables could be combined into the proposed
 282 framework.

283 For example, in symbolic data analysis, an interval-valued variable has special binary representa-
 284 tions such as “Lower-Upper” and “Center-Radius” (Billard and Diday, 2003; Sun et al., 2018). Linear
 285 regression can then be conducted on these binary numeric variables (Wei et al., 2017), with the random
 286 effects incorporated analogously. Similarly, we can also formulate the regressions on other symbolic
 287 variables in symbolic data analysis, histograms, and distribution functions, with more complicated
 288 characteristics based on the Wasserstein distance (Irpino and Verde, 2015), and consider the related
 289 random effects to extend them to the proposed framework. Finally, some theoretical properties for
 290 the random effects associated with diversified variables, such as Remarks 1–3, remain to be checked,
 291 which need further research in the future.

292 Next, the introduction of random effects promotes the performance of linear regression for longi-
 293 tudinal complex data, while the complexity of random effects leads to an extra cost of computation
 294 and a loss of degrees of freedom. Thus, the trade-off between the improvement in fitting accuracy
 295 and complexity of random effects is worthy of consideration, which falls into the suitable selection of
 296 random effects. As an important issue for the LMM, many statistical solutions for traditional scalar
 297 covariates have been proposed, such as the Bayesian information criteria selector (Fitzmaurice et al.,
 298 2011) and joint selection (Bondell et al., 2010). Furthermore, we can determine the constitution of
 299 the random effects from the practical and empirical perspectives (e.g., some financial knowledge in
 300 the real data study). We can also conduct a series of alternative CompLMM associated with all the
 301 possible constitutions of the random effects, including CompLM, and select the balanced one that
 302 approximates the best improvement with relatively few random effects.

303 4 Numerical experiment

304 In this section, we report the simulation results to evaluate the performance of the proposed param-
 305 eter estimation for CompLMM. Three measures are introduced: the squared ratio error (SRE) for
 306 scalar responses, integral squared error (ISE) for functions, and absolute percentage error (APE) for

307 compositions. These are respectively defined in Model (1) as

$$\begin{aligned} \text{SRE} &= \sum_{i=1}^N \sum_{j=1}^{n_i} (y_{ij} - \hat{y}_{ij})^2 / \sum_{i=1}^N \sum_{j=1}^{n_i} y_{ij}^2, \\ \text{ISE}(\hat{\beta}_k) &= d_{\mathcal{L}^2}^2(\beta_k, \hat{\beta}_k) \quad (k = 1, 2, \dots, p_\mu), \\ \text{APE}(\hat{\gamma}_k) &= d_a(\gamma_k, \hat{\gamma}_k) / \|\gamma_k\|_a \times 100\% \quad (k = 1, 2, \dots, p_c), \end{aligned}$$

308 where \hat{y}_{ij} , $\hat{\mu}_{ijk}$, and $\hat{\mathbf{c}}_{ijk}$ denote the related fitted values. Specifically, a lower SRE, ISE, or APE value
309 indicates a more accurate fitting for the specific response, function, or composition, respectively.

310 We generate the data from Model (1) as

$$y_{ij} = 2 + \alpha_1 x_{ij1} + \int_0^1 \beta_1 \mu_{ij1} + \int_0^1 \beta_2 \mu_{ij2} + (\gamma_1, \mathbf{c}_{ij1})_a + (\gamma_2, \mathbf{c}_{ij2})_a + a_{i0} + \int_0^1 b_{i1} \nu_{ij1} + (\mathbf{r}_{i1}, \mathbf{w}_{ij1})_a + \varepsilon_{ij},$$

311 where seven three-order B-spline basis functions defined by four equally spaced interior knots over
312 $[0, 1]$, say $\phi = \{\phi_1, \phi_2, \dots, \phi_7\}$, and the ilr coordinates from (7) and (8) are adopted. The detailed
313 parameter settings are introduced as follows.

314 a) In the fixed effects, x_{ij1} is independently generated from the standard normal distribution, with
315 $\alpha_1 = 5$; β_1 and β_2 are linearly combined by ϕ , with the symmetric combination coefficients.
316 That is, for any $t \in [0, 1]$,

$$\beta_1(t) = \sum_{j=1}^7 (4-j)\phi_j(t) \quad \text{and} \quad \beta_2(t) = \sum_{j=1}^7 (j-4)\phi_j(t);$$

317 respectively; μ_{ij1} and μ_{ij2} are described as $n = 200$ samples observed at times $\{t_1, t_2, \dots, t_n\}$
318 from the linear combinations of ϕ with measurement errors, that is,

$$\mu_{ijk}(t_l) = \mathbf{u}'_{ijk} \phi(t_l) + \epsilon_{ijkl} \quad (k = 1, 2; l = 1, 2, \dots, n),$$

319 where the expansion coefficients of both functions are sampled from $\mathcal{N}(\mathbf{0}_7, \mathbf{I}_7)$, and the errors
320 are generated from $\mathcal{N}(0, 0.1^2)$; \mathbf{c}_{ij1} and \mathbf{c}_{ij2} are separately generated from the simplicial normal
321 distribution $\mathcal{N}_S(\mathbf{0}_2, \mathbf{I}_2)$ (Mateu-Figueras et al., 2013), with the compositional coefficients $\gamma_1 =$
322 $(0.8, 0.1, 0.1)'$ and $\gamma_2 = (0.2, 0.2, 0.6)'$ in S^3 , respectively.

323 b) In the random effects, the covariates are constituted by the intercept a_{i0} and the first function
324 and composition, namely $\nu_{ij1} = \mu_{ij1}$ and $\mathbf{w}_{ij1} = \mathbf{c}_{ij1}$; b_{i1} and \mathbf{r}_{i1} are represented by the ex-
325 pansion coefficients under ϕ and ilr coordinates, say $\boldsymbol{\theta}_{i1}$ and \mathbf{r}_{i1}^* , respectively. The parameters
326 involved, namely $\boldsymbol{\pi}_i = (a_{i0}; \boldsymbol{\theta}_{i1}; \mathbf{r}_{i1}^*)$, are then jointly generated from $\mathcal{N}(\mathbf{0}_{10}, \mathbf{G})$, where \mathbf{G} is
327 blocked diagonal, namely $\mathbf{G} = \text{diag}(9, \mathbf{G}_{\theta^*}, 0.5\mathbf{I}_4, \mathbf{G}_r)$ with

$$\mathbf{G}_{\theta^*} = \begin{pmatrix} 9 & 4.8 & 0.6 \\ 4.8 & 4 & 1 \\ 0.6 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{G}_r = \begin{pmatrix} 9 & 4.8 \\ 4.8 & 4 \end{pmatrix}.$$

328 c) ε_{ij} is independently generated from $\mathcal{N}(0, \sigma^2)$, where σ takes a value of 0.5, 1, 1.5, or 3 to reflect
 329 the signal-to-noise ratio (SNR) from strong to weak.

330 Three combinations of the number of individuals N and sample size for each individual n_i are con-
 331 sidered: $(N, n_i) = (100, 60)$, $(300, 60)$, and $(300, 90)$. For each case, we independently replicate the
 332 simulation 500 times and conduct the proposed CompLMM as well as the baseline CompLM to pro-
 333 vide a comparison. Table 1 summarizes the general performances of the two models for the SNRs
 334 across the sample sizes. Table 2 reports the estimated results for the functional and compositional
 335 coefficients of the two models with $(N, n_i) = (100, 60)$, and those with the other settings of (N, n_i) are
 336 reported in the Appendix. Moreover, Fig. 1 visualizes the specific curves of the estimated functional
 337 coefficients in randomly selected replications.

338 As shown in Table 1, the estimated coefficients for the scalar covariates (including the inter-
 339 cept) obtained from both CompLMM and CompLM on average approximate the related ideal values,
 340 while those from CompLMM are more stable with lower standard deviations than the baseline: 0.013
 341 (CompLMM) vs. 0.105 (CompLM) for $\hat{\alpha}_1$ with $(N, n_i) = (100, 60)$ and $\sigma = 0.5$. For the functions,
 342 CompLMM sharply improves the estimation efficiency of the function-type coefficients, for which both
 343 the means and the standard deviations of the ISE are multiple times lower than those from CompLM:
 344 0.03 and 0.021 (CompLMM) vs. 1.165 and 0.953 (CompLM). Moreover, the two methods perform the
 345 same for the compositions, where the ISE values for CompLMM are slightly more stable than those
 346 for CompLM. Finally, CompLMM estimates σ well for the SNRs, **whereas the estimates of CompLM**
 347 **far exceed the corresponding true values**. When noise is extremely large (i.e., $\sigma = 3$), the related
 348 estimate may be biased in a poor sample (e.g., $(N, n_i) = (100, 60)$), whereas such bias is mitigated if
 349 the sample is sufficiently large. Moreover, CompLM almost fails to fit the responses, since the values
 350 of the SRE are on average above 0.4 when $\sigma \leq 1.5$; by contrast, **for CompLMM, the average values of**
 351 **the SRE are significantly low and close to 0**.

352 The aforementioned conclusions on the estimated parameters for the functional and compositional
 353 covariates are confirmed by the expansion coefficients for the functions and detailed estimates for the
 354 compositions, as shown in Table 2. Specifically, the average estimates of the functions and compositions
 355 using both CompLMM and CompLM approach the related ideal values for the SNRs. However, the
 356 relatively high standard deviations for the expansion coefficients from CompLM lead to nonsignificant
 357 regression results. For example, the test statistic for $\hat{\lambda}_{11}$ with $\sigma = 0.5$, simply measured by $2.946/3.154$,
 358 is less than the threshold value at the significance level of 0.05 (or even larger), which implies that we
 359 cannot reject the null hypothesis $H_0 : \lambda_{11} = 0$. By contrast, the same test statistic from CompLMM,
 360 similarly measured by $2.962/0.485$, is more than the threshold value at the 0.05 level (or even smaller).
 361 For the compositions, owing to the limited magnitude of the proportions inside, the two methods show

Table 1: Means and standard derivations (in brackets) of the three measures and estimated parameters for the scalar covariates and errors. The ideal values of the intercept and $\hat{\alpha}_1$ are 2 and 5, respectively and those of $\hat{\sigma}^2$ coincide with the true setting of σ .

(N, n_i)	Model	Scalar		Functional		Compositional		$\hat{\sigma}^2$	SRE
		Intercept	$\hat{\alpha}_1$	ISE($\hat{\beta}_1$)	ISE($\hat{\beta}_2$)	APE($\hat{\gamma}_1$)	APE($\hat{\gamma}_2$)		
$\sigma = 0.5$									
(100, 60)	CompLMM	1.97 (0.292)	5 (0.013)	0.03 (0.021)	0.016 (0.012)	6.185 (0.269)	0.448 (0.026)	0.263 (0.02)	0.016 (0.011)
	CompLM	1.971 (0.297)	4.996 (0.105)	1.165 (0.953)	1.194 (0.819)	6.177 (0.287)	0.447 (0.032)	22.314 (2.176)	0.425 (0.032)
(300, 60)	CompLMM	2.021 (0.176)	5 (0.007)	0.009 (0.007)	0.005 (0.004)	6.227 (0.164)	0.452 (0.016)	0.25 (0.003)	0.009 (0.003)
	CompLM	2.02 (0.177)	4.994 (0.06)	0.416 (0.299)	0.352 (0.244)	6.23 (0.168)	0.454 (0.018)	22.284 (1.178)	0.427 (0.019)
(300, 90)	CompLMM	2.007 (0.168)	5 (0.005)	0.007 (0.006)	0.003 (0.002)	6.238 (0.172)	0.453 (0.017)	0.25 (0.002)	0.01 (0.004)
	CompLM	2.005 (0.17)	5.002 (0.054)	0.275 (0.217)	0.243 (0.177)	6.24 (0.176)	0.454 (0.019)	22.288 (1.332)	0.427 (0.021)
$\sigma = 1$									
(100, 60)	CompLMM	1.97 (0.293)	4.999 (0.024)	0.078 (0.054)	0.065 (0.049)	6.184 (0.027)	0.448 (0.026)	1.067 (0.08)	0.035 (0.011)
	CompLM	1.972 (0.298)	4.996 (0.107)	1.204 (0.977)	1.234 (0.855)	6.177 (0.287)	0.447 (0.032)	23.062 (2.179)	0.417 (0.032)
(300, 60)	CompLMM	2.021 (0.176)	5 (0.013)	0.023 (0.016)	0.02 (0.014)	6.227 (0.164)	0.452 (0.016)	0.998 (0.012)	0.027 (0.003)
	CompLM	2.02 (0.177)	4.994 (0.061)	0.433 (0.313)	0.364 (0.249)	6.23 (0.168)	0.454 (0.018)	23.033 (1.179)	0.419 (0.018)
(300, 90)	CompLMM	2.007 (0.168)	5 (0.011)	0.016 (0.011)	0.013 (0.01)	6.238 (0.172)	0.453 (0.017)	0.999 (0.009)	0.028 (0.004)
	CompLM	2.005 (0.17)	5.002 (0.055)	0.281 (0.22)	0.252 (0.183)	6.239 (0.176)	0.454 (0.019)	23.038 (1.332)	0.419 (0.02)
$\sigma = 1.5$									
(100, 60)	CompLMM	1.971 (0.294)	4.998 (0.038)	0.162 (0.112)	0.147 (0.109)	6.183 (0.27)	0.448 (0.027)	2.436 (0.197)	0.066 (0.012)
	CompLM	1.972 (0.298)	4.995 (0.11)	1.27 (1.018)	1.3 (0.912)	6.176 (0.287)	0.447 (0.032)	24.309 (2.183)	0.404 (0.031)
(300, 60)	CompLMM	2.021 (0.176)	5 (0.02)	0.046 (0.033)	0.044 (0.032)	6.227 (0.164)	0.452 (0.016)	2.251 (0.032)	0.055 (0.003)
	CompLM	2.02 (0.177)	4.994 (0.063)	0.459 (0.333)	0.385 (0.261)	6.23 (0.168)	0.454 (0.018)	24.281 (1.18)	0.406 (0.018)
(300, 90)	CompLMM	2.007 (0.168)	5.001 (0.016)	0.031 (0.022)	0.029 (0.022)	6.238 (0.172)	0.453 (0.017)	2.249 (0.02)	0.056 (0.004)
	CompLM	2.005 (0.17)	5.002 (0.056)	0.292 (0.227)	0.268 (0.194)	6.239 (0.176)	0.454 (0.019)	24.288 (1.333)	0.406 (0.02)
$\sigma = 3$									
(100, 60)	CompLMM	1.973 (0.297)	4.995 (0.076)	0.605 (0.416)	0.579 (0.414)	6.181 (0.273)	0.448 (0.028)	9.882 (0.826)	0.206 (0.022)
	CompLM	1.973 (0.302)	4.994 (0.125)	1.629 (1.26)	1.661 (1.195)	6.174 (0.289)	0.447 (0.033)	31.039 (2.212)	0.347 (0.027)
(300, 60)	CompLMM	2.021 (0.177)	5 (0.04)	0.169 (0.128)	0.174 (0.126)	6.227 (0.165)	0.452 (0.017)	9.052 (0.159)	0.183 (0.007)
	CompLM	2.02 (0.178)	4.994 (0.071)	0.59 (0.427)	0.504 (0.341)	6.23 (0.169)	0.454 (0.019)	31.022 (1.193)	0.348 (0.016)
(300, 90)	CompLMM	2.008 (0.169)	5.001 (0.031)	0.111 (0.084)	0.114 (0.087)	6.237 (0.173)	0.453 (0.018)	9.014 (0.097)	0.186 (0.007)
	CompLM	2.006 (0.171)	5.002 (0.061)	0.359 (0.271)	0.351 (0.246)	6.238 (0.177)	0.454 (0.019)	31.036 (1.34)	0.349 (0.017)

Table 2: Means and standard derivations (in brackets) of the estimated expansion coefficients for the functions and re-transformed coefficients for compositions with $(N, n_i) = (100, 60)$. The ideal values of $\hat{\lambda}_k = (\hat{\lambda}_{k1}, \hat{\lambda}_{k2}, \dots, \hat{\lambda}_{k7})'$ ($k = 1, 2$) are $\hat{\lambda}_1 = (3, 2, \dots, -3)'$ and $\hat{\lambda}_2 = (-3, -2, \dots, 3)'$, and those of $\hat{\gamma}_k$ are γ_k ($k = 1, 2$).

Model	Coefficients		Functional							Compositional		
	λ_k / γ_k	$\hat{\lambda}_{k1}$	$\hat{\lambda}_{k2}$	$\hat{\lambda}_{k3}$	$\hat{\lambda}_{k4}$	$\hat{\lambda}_{k5}$	$\hat{\lambda}_{k6}$	$\hat{\lambda}_{k7}$	$\hat{\gamma}_{k1}$	$\hat{\gamma}_{k2}$	$\hat{\gamma}_{k3}$	
$\sigma = 0.5$												
CompLMM	$k = 1$	2.962 (0.485)	1.996 (0.509)	1.003 (0.401)	0.008 (0.419)	-1.026 (0.564)	-1.989 (0.599)	-3.004 (0.472)	0.786 (0.055)	0.101 (0.018)	0.112 (0.041)	
	$k = 2$	-2.947 (0.377)	-2.047 (0.468)	-0.966 (0.4)	-0.031 (0.415)	1.05 (0.553)	1.936 (0.581)	3.039 (0.459)	0.2 (0.001)	0.2 (0.001)	0.6 (0.002)	
CompLM	$k = 1$	2.946 (3.154)	2.063 (3.848)	0.991 (3.239)	-0.059 (3.412)	-0.841 (4.609)	-2.246 (5.008)	-2.719 (4.059)	0.785 (0.059)	0.103 (0.021)	0.113 (0.043)	
	$k = 2$	-2.755 (3.141)	-2.297 (3.892)	-0.806 (3.312)	-0.181 (3.593)	1.272 (4.908)	1.964 (5.147)	2.78 (4.095)	0.2 (0.013)	0.199 (0.012)	0.6 (0.017)	
$\sigma = 1$												
CompLMM	$k = 1$	2.969 (0.804)	1.981 (0.931)	1.011 (0.753)	0.01 (0.806)	-1.029 (1.102)	-2.008 (1.19)	-2.987 (0.934)	0.786 (0.055)	0.102 (0.018)	0.112 (0.041)	
	$k = 2$	-2.911 (0.767)	-2.072 (0.961)	-0.953 (0.816)	-0.044 (0.835)	1.09 (1.098)	1.868 (0.159)	3.086 (0.924)	0.2 (0.003)	0.2 (0.003)	0.6 (0.004)	
CompLM	$k = 1$	2.927 (3.204)	2.078 (3.926)	0.981 (3.306)	-0.044 (3.473)	-0.864 (4.683)	-2.233 (5.087)	-2.725 (4.146)	0.785 (0.059)	0.103 (0.021)	0.113 (0.043)	
	$k = 2$	-2.716 (3.17)	-2.339 (3.938)	-0.775 (3.371)	-0.206 (3.654)	1.306 (4.994)	1.92 (5.249)	2.812 (4.178)	0.2 (0.013)	0.199 (0.012)	0.6 (0.018)	
$\sigma = 1.5$												
CompLMM	$k = 1$	2.988 (1.161)	1.962 (1.379)	1.032 (1.129)	-0.004 (1.216)	-1 (1.673)	-2.058 (1.808)	-2.956 (1.411)	0.786 (0.055)	0.102 (0.018)	0.112 (0.041)	
	$k = 2$	-2.889 (1.167)	-2.089 (1.456)	-0.951 (1.233)	-0.049 (1.255)	1.118 (1.644)	1.81 (1.725)	3.128 (1.386)	0.2 (0.005)	0.2 (0.004)	0.6 (0.006)	
CompLM	$k = 1$	2.909 (3.288)	2.094 (4.043)	0.97 (3.406)	-0.03 (3.57)	-0.888 (4.807)	-2.22 (5.221)	-2.731 (4.275)	0.784 (0.059)	1.03 (0.021)	0.113 (0.043)	
	$k = 2$	-2.676 (3.236)	-2.381 (4.03)	-0.745 (3.466)	-0.231 (3.749)	1.34 (5.126)	1.876 (5.399)	2.845 (4.299)	0.2 (0.013)	0.199 (0.013)	0.6 (0.018)	
$\sigma = 3$												
CompLMM	$k = 1$	3.022 (2.274)	1.952 (2.726)	1.04 (2.252)	0.009 (2.405)	-1.025 (3.346)	-2.076 (3.618)	-2.937 (2.796)	0.785 (0.056)	0.102 (0.019)	0.112 (0.041)	
	$k = 2$	-2.803 (2.277)	-2.167 (2.842)	-0.933 (2.411)	-0.031 (2.466)	1.123 (3.25)	1.733 (3.412)	3.197 (2.774)	0.2 (0.009)	0.2 (0.008)	0.6 (0.012)	
CompLM	$k = 1$	2.853 (3.716)	2.139 (4.6)	0.937 (3.873)	0.014 (4.053)	-0.96 (5.444)	-2.181 (5.907)	-2.748 (4.867)	0.784 (0.06)	0.103 (0.021)	0.113 (0.043)	
	$k = 2$	-2.559 (3.637)	-2.507 (4.544)	-0.653 (3.943)	-0.306 (4.221)	1.443 (5.762)	1.743 (6.103)	2.944 (4.861)	0.2 (0.015)	0.2 (0.014)	0.6 (0.021)	

362 no remarkable difference.

363 As exemplified in Fig. 1, the curves (in red) of the estimated functional coefficients from CompLMM
364 move closer to the true settings (in gray) than those (in cyan) from CompLM. For both the increasing
365 and the decreasing cases, CompLMM fits the functions well across the interval, whereas CompLM,
366 despite capturing the general trends of the functions, creates relatively large periodic perturbations.
367 Moreover, the biases between the true and fitted curves from the two models are eliminated gradually
368 as the sample size increases (e.g., $(N, n_i) = (300, 90)$).

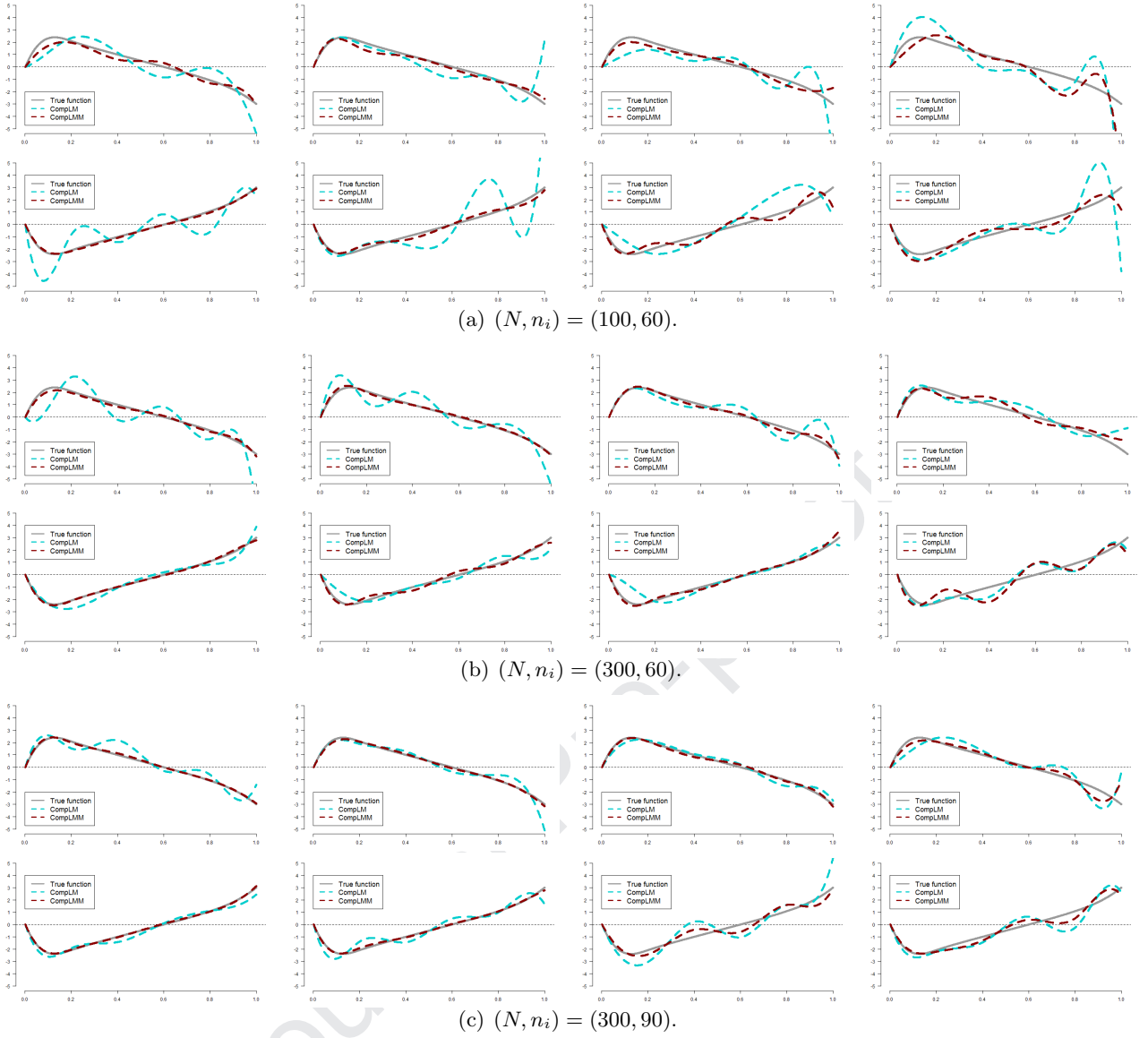


Fig. 1: Curves of the estimated functional coefficients. The columns from left to right denote the four levels of the SNRs from $\sigma = 0.5$ to $\sigma = 3$. The upper and lower sub-rows indicate the two functional covariates.

369 In summary, the proposed CompLMM succeeds in addressing the longitudinal features within
 370 complex data with diversified characteristics, especially those with functional characteristics.

371 5 Application

372 In this section, we adopt the proposed CompLMM in a real data study to demonstrate its usefulness.
 373 The existing approach to complex data modeling, namely CompLM (Wang et al., 2015, 2019a), is also
 374 used for comparison purposes.

375 Using the case of China’s stock market, we aim to measure the influence of indirect information on
 376 stock prices as well as the historical price trend. As exemplified by numerous studies, macroeconomic
 377 indicators (Chen et al., 1986), public online emotion (Ruan et al., 2018; Zhou et al., 2017), and
 378 analysts’ recommendations (Duan et al., 2013) may improve the interpretability and accuracy of
 379 models for this problem. In this case, we regress the daily closing price (DCP) of stocks against the
 380 related daily volume (DV), intraday percentage return (IPR), and online investors’ emotions (OIE)
 381 in the former session. Data on the constituent stocks in CSI300 from January 8 to April 29, 2016
 382 (75 trading days) are collected from the Wind service. Stocks that have fewer than 40 active trading
 383 days are omitted, and finally 271 stocks are left. Specifically, both the DCP and the DV are scalar,
 384 and the IPR recorded every five minutes is described as a smoothing curve from opening to closing.
 385 Moreover, the data on the OIE are measured by Zhou et al. (2017), where observations are naturally
 386 of a compositional structure associated with five types of emotions labeled “Anger,” “Disgust,” “Joy,”
 387 “Sadness,” and “Fear.” Thus, the regression contains two scalar covariates (including the intercept),
 388 one functional covariate, and one compositional covariate.

389 Then, we conduct CompLMM with all four covariates in the random effects as well as CompLM.
 390 Specifically, the IPR on each trading day is separately represented by seven expansion coefficients
 391 under the B-spline basis functions ϕ described in Section 4 over $[0, 1]$, where 0 and 1 indicate the
 392 opening (9:30 a.m.) and closing (15:00 p.m.) times, respectively. CompLM shows a poor result in
 393 this regression, as the variance of the residuals from it reaches an unacceptable level (460.43). By
 394 contrast, the introduction of the random effects in CompLMM reduces that variance to only 4.02,
 395 which makes for a reliable interpretation. Table 3 reports the estimated results for the fixed effects
 396 from the two models and Fig. 2 presents the related curves and pie charts of the estimated functional
 397 and compositional coefficients, respectively.

398 As shown in Table 3, the contribution of the DV to the stock price is contrasting in the two
 399 models: positive (0.09) in CompLMM and negative (-0.13) in CompLM. However, the absolute values
 400 of both coefficients are low, implying that the DV may not have a significant influence on the stock
 401 price. For the IPR, the directions of the estimated expansion coefficients from the two models are
 402 close in general, with six of the seven components having a consistent sign. As displayed in the left
 403 column of Fig. 2, the two images of the functional coefficients share a similar shape and the returns
 404 near 10:30 a.m., 14:00 p.m., and the closing time show relatively high marginal effects on the stock
 405 price. The difference between the two curves is that the range of values in CompLM is around 10 times
 406 larger than that in CompLMM, which accounts for the bad performance of CompLM to a great extent.
 407 Finally, as presented in the right column of Fig. 2, the two models differ in the estimated compositional
 408 coefficient for the OIE, although they both consider “Joy” and “Fear” to be two important emotions

409 for explaining the DCP. In CompLMM, “Joy” has the largest influence on the stock price (proportion
 410 of 0.66), with “Fear” second (0.21); however, these two emotions change places in CompLM: 0.21 for
 411 “Joy” vs. 0.57 for “Fear”. Since the increase in the inner part of a composition implies a general
 412 decrease in the others, it is hard to measure the influence of a specific part separately (Pawlowsky-
 413 Glahn et al., 2015). Hence, we only briefly discuss the marginal contribution of each type of emotion
 414 in the regression.

Table 3: Estimated coefficients for the fixed effects in the real data study. The estimated functional coefficient for the IPR is reported by its expansion coefficients, as indicated by the sub-columns ϕ_j ($j = 1, 2, \dots, 7$).

Model	Intercept	DV	IPR							Anger	Disgust	Joy	Sadness	Fear
			ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7					
CompLMM	15.81	0.09	-5.58	10.24	-7	1.13	11.11	-15.49	9.58	0.03	0.06	0.66	0.04	0.21
CompLM	21.29	-0.13	-84.56	130.84	-29.2	-74.13	199.59	-233.4	137.69	0.02	0.15	0.21	0.05	0.57

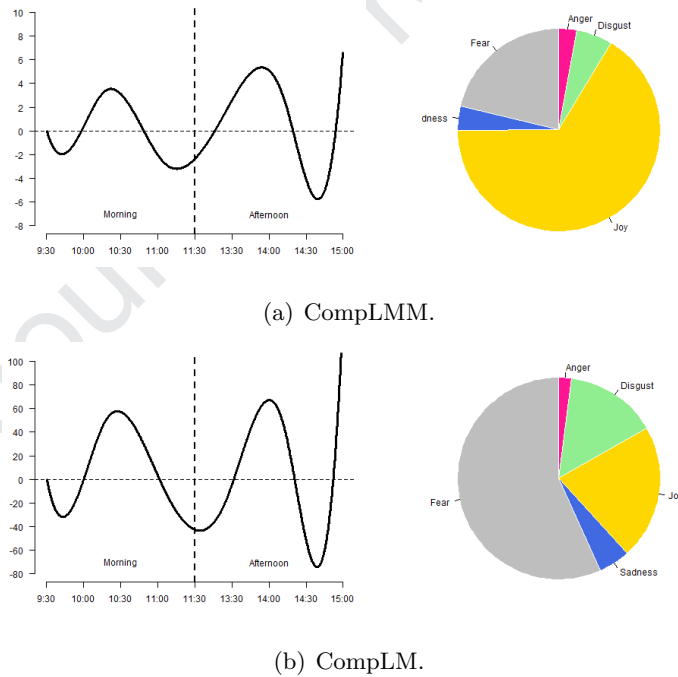


Fig. 2: Curves and pie charts of the estimated functional and compositional coefficients for the IPR and OIE, respectively. The vertical dotted line in the curve divides the trading day into morning and afternoon.

415 Next, we focus on the estimated results for the random effects for CompLMM, as reported in
 416 Table 4. The large variance of the intercept (314.3) indicates that the stocks involved have great

417 differences in prices. The variance of the DV is relatively small (i.e., only 0.1), which implies that its
 418 influence has few changes across stocks; therefore, the DV can be regarded as an inessential factor in
 419 this case. To describe the overall variation of the functional coefficient, we plot the variance function
 420 based on the covariance matrix of the seven expansion coefficients, where the point-wise variances
 421 during all trading hours exceed 50 in general, especially those before 10:00 a.m. and after 14:30 p.m.
 422 This result verifies that past trends of the return from different stocks have different influences on their
 423 prices in the future. Finally, we sum the variances of the four ilr coordinates and obtain the related
 424 total variance of the compositional covariate for the OIE as 36.3. This result verifies that indirect
 425 information such as the OIE, although shared by all stocks, can also enhance the performance of the
 426 regression model for the stock price in various ways.

Table 4: Estimated covariance matrix of the random effects for CompLMM in the real data study. The sub-columns ϕ_j ($j = 1, 2, \dots, 7$) are the same as in Table 3 and w_{1k}^* ($k = 1, 2, \dots, 4$) indicate the ilr coordinates of the compositional coefficient for the OIE. The variances are highlighted in bold. The variance function and total covariance of the functional and compositional coefficients for the IPR and OIE are also plotted and reported.

	Intercept DV		IPR							OIE			
			ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	w_{11}^*	w_{12}^*	w_{13}^*	w_{14}^*
Intercept	314.6	2.9	60.8	-73.6	20.8	38.4	-63.4	163.4	40.6	15.3	-1.3	-24.6	1.2
DV		0.1	0.5	-0.8	0.6	-0.2	-0.6	1.3	-1	0.7	0.2	-0.6	0.4
ϕ_1			669.9	-743.6	442.9	-282.7	245.5	-195.4	106.8	3.2	4.3	-8	1.2
ϕ_2				1039.1	-761.5	562.9	-533.9	418.3	-233.6	-6.5	-2.7	12.8	-4.5
ϕ_3					693.1	-608.3	599.1	-444.7	239.2	6.1	1.7	-9.6	8.1
ϕ_4						680.1	-776.4	586.6	-306.5	-3.7	-1.9	4.5	-10.5
ϕ_5							1152.1	-1151.5	727.9	-8.9	-0.4	13.6	-5.3
ϕ_6								1637.4	-1348.7	24.6	7.8	-31.3	30.7
ϕ_7									1362.5	-23.2	-11.8	25	-31.9
w_{11}^*										7.1	2.8	-8.8	7.5
w_{12}^*											5.6	-2.3	2.8
w_{13}^*												13.2	-10.1
w_{14}^*													10.4

Variance function for IPR

Total variance for OIE: 36.3

427 In conclusion, the real data study illustrates the potential of the proposed CompLMM for lon-
 428 gitudinal complex data from an application perspective. Introducing random effects containing the
 429 scalar, functional, and compositional covariates, our method measures the subject-specific character-
 430 istics of each stock and improves the performance of the regression on diversified types of variables

431 from different sources. However, there remains some problems to be discussed from both theoretical
432 and practical perspectives, such as a more exhaustive explanation of the functional or compositional
433 coefficients and the choice of direct and indirect indicators for China's stock market. These issues
434 need to be addressed in future work.

435 **6 Discussion**

436 This study investigates an LMM technique for longitudinal complex data named CompLMM, involv-
437 ing scalar continuous response and complex data covariates with diversified characteristics. Through
438 random effects that describe the differences across individuals, CompLMM can extract further in-
439 formation from the residuals obtained by the existing linear model for complex data and shows a
440 significant improvement in fitting responses. Following the linear framework of complex data model-
441 ing, CompLMM first unifies the numeric representation of different types of variables such that the
442 traditional LMM can then be conducted to obtain the intermediate results and transform them back
443 to have related diversified features. This model also encourages a more comprehensive interpreta-
444 tion for regression on complex data. Moreover, some theoretical properties are also presented that
445 support the computational procedure of the parameter estimation for CompLMM. As illustrated by
446 both the numerical experiment and the real data study, the proposed CompLMM succeeds in dealing
447 with longitudinal complex data and efficiently estimating the parameters with more reliable response
448 fittings.

449 We focus on the parameter estimation and its general interpretation for the proposed CompLMM.
450 However, the trade-off between the accuracy and interpretation of the proposed model also needs due
451 consideration, to which many solutions for traditional scalar covariates have been proposed. These
452 statistical methods provide instructive strategies for selecting random effects with diversified charac-
453 teristics, which face great challenges in theory but deserve further research. Meanwhile, practical and
454 empirical ways of determining the random effects also demand investigation. Moreover, many types
455 of complex data, as discussed in Section 3.3, have the potential to be modeled under the proposed
456 framework using related representations. The processing of these variables has been adopted by many
457 studies, but some of the theoretical properties for this study need detailed checks in the future.

458 Finally, the statistical inferences for multiple types of complex data, with functional, composi-
459 tional, and other more complicated features, are also an important and challenging issue in regression.
460 Although empirical methods (e.g., the bootstrap) have offered partial solutions to this problem, related
461 hypothesis tests for complex data such as the function with an infinite dimension and composition
462 involving constraints, should also be developed.

463 **Appendix: More results from the numerical experiment**

464 See Tables 5 and 6.

Table 5: Means and standard derivations (in brackets) of the estimated expansion coefficients for the functions and re-transformed coefficients for the compositions with $(N, n_i) = (300, 60)$. The ideal values are the same as in Table 2.

Model	Coefficients	Functional							Compositional		
	β_k / γ_k	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	γ_{k1}	γ_{k2}	γ_{k3}
$\sigma = 0.5$											
CompLMM	$k = 1$	2.988 (0.257)	2.011 (0.287)	0.983 (0.213)	0.004 (0.226)	-1.003 (0.308)	-2.007 (0.325)	-2.971 (0.25)	0.795 (0.033)	1.01 (0.012)	0.104 (0.023)
	$k = 2$	-3.009 (0.216)	-1.985 (0.269)	-1.015 (0.222)	0.02 (0.231)	0.982 (0.311)	2.018 (0.32)	2.973 (0.248)	0.2 (0.001)	0.2 (0.001)	0.6 (0.001)
CompLM	$k = 1$	3.257 (0.1956)	1.675 (2.439)	1.241 (1.984)	-0.257 (2.019)	-0.674 (2.701)	-2.224 (2.942)	-2.861 (2.369)	0.796 (0.034)	0.101 (0.013)	0.103 (0.024)
	$k = 2$	-3.039 (1.751)	-1.895 (2.136)	-1.121 (1.76)	0.172 (1.836)	0.774 (2.506)	2.195 (2.609)	2.885 (2.132)	0.199 (0.007)	0.2 (0.007)	0.6 (0.01)
$\sigma = 1$											
CompLMM	$k = 1$	2.972 (0.424)	2.024 (0.526)	0.974 (0.417)	0.01 (0.447)	-1.007 (0.608)	-2.019 (0.637)	-2.95 (0.49)	0.795 (0.033)	0.101 (0.012)	0.2 (0.002)
	$k = 2$	-3.02 (0.427)	-1.973 (0.53)	-1.028 (0.436)	0.036 (0.457)	0.969 (0.615)	2.032 (0.634)	2.955 (0.494)	0.2 (0.002)	0.2 (0.001)	0.6 (0.002)
CompLM	$k = 1$	3.24 (2.004)	1.688 (2.499)	1.232 (2.03)	-0.251 (2.065)	-0.679 (2.762)	-2.233 (3.001)	-2.843 (2.421)	0.796 (0.034)	0.101 (0.013)	0.103 (0.024)
	$k = 2$	-3.044 (1.776)	-1.887 (2.17)	-1.131 (1.787)	0.186 (1.866)	0.763 (2.562)	2.209 (2.664)	2.868 (2.161)	0.199 (0.007)	0.2 (0.007)	0.6 (0.01)
$\sigma = 1.5$											
CompLMM	$k = 1$	2.956 (0.609)	3.224 (2.068)	1.701 (2.581)	1.224 (2.094)	-0.245 (2.133)	-0.684 (2.852)	-2.241 (3.09)	0.796 (0.034)	0.101 (0.013)	0.104 (0.023)
	$k = 2$	-3.029 (0.637)	-1.962 (0.792)	-1.041 (0.652)	0.055 (0.684)	0.951 (0.923)	2.05 (0.95)	2.935 (0.741)	0.2 (0.002)	0.2 (0.002)	0.6 (0.003)
CompLM	$k = 1$	2.956 (0.609)	2.036 (0.771)	0.966 (0.623)	0.017 (0.667)	-1.011 (0.907)	-2.031 (0.948)	-2.93 (0.726)	0.795 (0.033)	0.101 (0.012)	0.104 (0.023)
	$k = 2$	-3.05 (1.825)	-1.88 (2.233)	-1.141 (1.838)	0.201 (1.922)	0.751 (2.65)	2.223 (2.753)	2.851 (2.216)	0.199 (0.007)	0.2 (0.007)	0.6 (0.011)
$\sigma = 3$											
CompLMM	$k = 1$	2.918 (1.183)	2.064 (1.515)	0.945 (1.241)	0.037 (1.317)	-1.025 (1.789)	-2.065 (1.869)	-2.864 (1.435)	0.795 (0.034)	0.101 (0.012)	0.104 (0.023)
	$k = 2$	-3.061 (1.261)	-1.924 (1.572)	-1.084 (1.297)	0.116 (1.364)	0.889 (1.836)	2.117 (1.879)	2.865 (1.458)	0.2 (0.005)	0.2 (0.004)	0.6 (0.007)
CompLM	$k = 1$	3.176 (2.348)	1.739 (2.938)	1.198 (2.383)	-0.227 (2.446)	-0.698 (3.276)	-2.266 (3.513)	-2.774 (2.822)	0.796 (0.035)	0.101 (0.013)	0.103 (0.024)
	$k = 2$	-3.067 (2.088)	-1.859 (2.572)	-1.172 (2.115)	0.244 (2.217)	0.717 (3.08)	2.264 (3.188)	2.8 (2.515)	0.199 (0.009)	0.2 (0.008)	0.6 (0.012)

Table 6: Means and standard derivations (in brackets) of the estimated expansion coefficients for the functions and re-transformed coefficients for the compositions with $(N, n_i) = (300, 90)$. The ideal values are the same as in Table 2.

Model	Coefficients β_k / γ_k	Functional							Compositional		
		ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	γ_{k1}	γ_{k2}	γ_{k3}
$\sigma = 0.5$											
CompLMM	$k = 1$	3.017 (0.246)	1.985 (0.224)	1.021 (0.179)	-0.021 (0.187)	-0.989 (0.246)	-2.003 (0.248)	-2.999 (0.198)	0.798 (0.035)	0.1 (0.011)	0.102 (0.026)
	$k = 2$	-2.974 (0.163)	-2.031 (0.2)	-0.976 (0.173)	-0.021 (0.188)	1.019 (0.256)	1.991 (0.269)	2.998 (0.209)	0.2 (0.001)	0.2 (0.001)	0.6 (0.001)
CompLM	$k = 1$	3.132 (1.58)	1.834 (1.964)	1.195 (1.675)	-0.211 (1.726)	-0.818 (2.298)	-2.1 (2.364)	-2.987 (1.86)	0.798 (0.036)	0.1 (0.012)	0.102 (0.026)
	$k = 2$	-3.018 (1.497)	-1.916 (1.752)	-1.093 (1.432)	0.077 (1.541)	0.893 (2.112)	2.137 (2.288)	3.022 (1.781)	0.199 (0.006)	0.2 (0.006)	0.6 (0.009)
$\sigma = 1$											
CompLMM	$k = 1$	3.03 (0.372)	1.967 (0.406)	1.043 (0.339)	-0.042 (0.361)	-0.975 (0.481)	-2.005 (0.484)	-3.002 (0.389)	0.797 (0.035)	0.1 (0.011)	0.102 (0.026)
	$k = 2$	-2.953 (0.324)	-2.062 (0.398)	-0.952 (0.345)	-0.043 (0.374)	1.039 (0.508)	1.98 (0.535)	3.001 (0.417)	0.2 (0.001)	0.2 (0.01)	0.6 (0.001)
CompLM	$k = 1$	3.145 (1.594)	1.817 (1.976)	1.216 (1.682)	-0.231 (1.734)	-0.804 (2.313)	-2.102 (2.377)	-2.989 (1.873)	0.798 (0.036)	0.1 (0.012)	0.102 (0.026)
	$k = 2$	-3.001 (1.518)	-1.942 (1.789)	-1.072 (1.474)	0.058 (1.586)	0.909 (2.16)	2.131 (2.329)	3.019 (1.817)	0.2 (0.006)	0.2 (0.006)	0.6 (0.009)
$\sigma = 1.5$											
CompLMM	$k = 1$	3.043 (0.513)	1.948 (0.597)	1.067 (0.501)	-0.064 (0.537)	-0.959 (0.717)	-2.009 (0.721)	-3.004 (0.581)	0.797 (0.035)	0.1 (0.011)	0.102 (0.026)
	$k = 2$	-2.931 (0.485)	-2.092 (0.596)	-0.929 (0.517)	-0.065 (0.56)	1.059 (0.759)	1.971 (0.799)	3.004 (0.624)	0.2 (0.002)	0.2 (0.002)	0.6 (0.003)
CompLM	$k = 1$	3.158 (1.621)	1.801 (2.007)	1.237 (1.704)	-0.25 (1.76)	-0.791 (2.35)	-2.104 (2.411)	-2.991 (1.904)	0.798 (0.036)	0.1 (0.012)	0.102 (0.026)
	$k = 2$	-2.984 (1.554)	-1.967 (1.846)	-1.051 (1.532)	0.04 (1.651)	0.926 (2.232)	2.124 (2.396)	3.016 (1.873)	0.2 (0.006)	0.2 (0.006)	0.6 (0.009)
$\sigma = 3$											
CompLMM	$k = 1$	3.086 (0.953)	1.885 (1.175)	1.142 (0.988)	-0.135 (1.062)	-0.906 (1.419)	-2.023 (1.423)	-3.009 (1.153)	0.797 (0.035)	0.1 (0.011)	0.102 (.026)
	$k = 2$	-2.872 (0.967)	-2.176 (1.191)	-0.863 (1.036)	-0.125 (1.117)	1.113 (1.511)	1.948 (1.586)	3.004 (1.244)	0.2 (0.004)	0.2 (0.004)	0.6 (0.006)
CompLM	$k = 1$	3.197 (1.778)	1.751 (2.198)	1.299 (1.855)	-0.309 (1.931)	-0.752 (2.586)	-2.109 (2.636)	-2.996 (2.099)	0.797 (0.036)	0.1 (0.012)	0.102 (0.026)
	$k = 2$	-2.932 (1.744)	-2.045 (2.112)	-0.989 (1.791)	-0.016 (1.931)	0.975 (2.577)	2.104 (2.731)	3.007 (2.143)	0.2 (0.004)	0.2 (0.004)	0.6 (0.006)

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