Zonotope-based interval estimation for discrete-time linear switched systems ⋆

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Abstract: This paper is concerned with the state interval estimation for discrete-time linear switched systems affected by unknown but bounded disturbances and measurement noises. A novel interval estimation method is proposed by integrating robust observer design with zonotopic techniques. By introducing L_∞ technique into observer design, the proposed method is effective in attenuating the influence of unknown disturbances and noises, and improving the accuracy of interval estimation. Based on the designed observer, the state interval estimation can be obtained by using zonotopic analysis. Finally, the feasibility and effectiveness of the proposed method are illustrated by numerical simulations.

Keywords: Zonotopic techniques, interval estimation, switched systems, observer design.

1. INTRODUCTION

Switched system, which is an important class of hybrid dynamical system, consists of continuous or discrete-time subsystems and a switching signal which determines the switching from one mode to another at every switching point. Switched system is an effective tool to describe a wide range of practical systems, including flight control systems (Vu and Morgansen, 2010) and network control systems (Donkers et al., 2011). Due to their powerful modelling capability, the stability analysis and control synthesis for switched systems have been extensively studied in the literature, see, e.g. Liberzon (2003); Zhao et al. (2012); Niu and Zhao (2013) and the references therein.

Apart from the stability analysis and controller design problems, state estimation is also very important for switched systems. In fact, state estimation has widely investigated in the control community such as fault diagnosis techniques (Wu et al., 2019), unknown input observer design (Guo et al., 2018) and robust controller design (Aslam et al., 2019). Therefore, many efforts have been devoted to state estimation for switched systems in the past decades. Specifically, Bejarano and Fridman (2011) designed an observer to ensure the reconstruction of the entire state in finite time for linear switched systems. For switched systems with Lipschitz nonlinear subsystems, a robust observer design method was proposed in Hernandez and García (2014) to realise state estimation. However, the existing results in Bejarano and Fridman (2011) and Hernandez and García (2014) are all state estimation for switched systems without uncertainties. In fact, the uncertainties such as process disturbances and measurement noises always exist in practical systems. To handle this problem, Bejarano and Pisano (2011) proposed the reduced-order unknown input switched observer to estimate the state of switched systems subjected to unknown inputs. For switched systems with both unknown disturbances and noises, Yang et al. (2017) designed a robust sliding-mode switched observer to estimate state with the aid of augmented system approach. However, these abovementioned methods all use the H_∞ technique to reduce the influence of unknown uncertainties and improve the estimation accuracy. Note that H_∞ norm is a measurement of energy-to-energy gain. As pointed out in Wang et al. (2017), the practical signals are not necessarily energy-bounded but have bounded peak values. Consequently, L_∞ norm, which aims to minimize the peak-to-peak gain, can be considered as an alternative solution to analyse the robustness performance of state estimation.

On the other hand, the results in Bejarano and Pisano (2011) and Yang et al. (2017) are all point-estimation of state and may not converge to the real state due to the existence of model and/or signals uncertainties. Therefore, state interval estimation approaches based on interval observer and zonotopic techniques get more attention for uncertain control systems in recent years. The basic idea

⋆ This work was partially supported by National Natural Science Foundation of China (61773145, 61973098, 61703296) and the Fund from the National Defense Key Discipline of Space Exploration, Landing and Reentry in Harbin Institute of Technology under Grant HIT.KLOF.2018.073.

Manuscript 1087 submitted to 21st IFAC World Congress, 2020.
Received November 7, 2019.

of interval observer is to design two sub-observers such that their estimation error dynamics are both cooperative and stable. The two sub-observers can provide the upper and lower bounds of the real system states. During the past decade, some interval observer design works have been devoted to various classes of linear or nonlinear regular systems (Raïssi et al., 2012; Efimov and Raïssi, 2016; Meslem et al., 2018). Specially, Ethabet et al. (2017) addressed the interval observer design issue for continuous-time linear switched systems. The interval observer design methods for discrete-time linear switched systems have been proposed in Dinh et al. (2019). However, it is not a trivial work to construct a cooperative and stable error system, and even impossible for some dynamical systems. Although the cooperative constraint can be relaxed by coordinate transformations, it may cause additional conservatism and limit the estimation accuracy. Meanwhile, the zonotope-based interval estimation methods can achieve a good trade-off between estimation accuracy and computation complexity, and have gained much attention by many researchers (Tang et al., 2019). Especially, Alamo et al. (2005) proposed a guaranteed state estimation method for nonlinear systems based on zonotopic techniques. Le et al. (2013) presented a new approach for guaranteed state estimation via zonotopes for uncertain multivariable systems. For time-varying dynamics systems with measurement uncertainties, Combastel (2015) designed a zonotopic Kalman observer to achieve the state interval estimation. In addition, some criteria such as P-radius (Le et al., 2013) and F-radius (Combastel, 2015) to decrease the size of zonotopes have been used to improve the accuracy of interval estimation. However, these methods by using zonotopic technique are all the state estimation for state-space systems. The case of zonotope-based interval estimation for switched systems has not been fully considered in the literature.

Motivated by the above-mentioned discussion, this paper proposes a novel state interval estimation method for discrete-time linear switched systems with unknown but bounded disturbances and noises. The main contributions of this paper are summarized as follows:

1) A novel interval estimation method that combines the observer design with zonotopic technique is proposed for discrete-time linear switched systems. The observer design conditions can be converted into linear matrix inequalities (LMIs).
2) By using zonotope approach, the proposed method can overcome the cooperative constraints used in the design of interval observer.
3) With the L_∞ technique used to reduce the influence of unknown disturbances and noises, the proposed method provides a systematic way to improve the accuracy of interval estimation.

2. PRELIMINARIES

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are the n and $m\times n$ dimensional Euclidean space, respectively. $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$, I_n denotes identity matrix with dimensions of n and 0 represents zero number, vector or matrix of appropriate dimensions. For a matrix $M \in \mathbb{R}^{m\times n}$, M^T denotes its transpose and $M(i,j)$ represents the element of M in the i-th row and the j-th column. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and $x(i)$ is the i-th component of x. The comparison operators $\geq$ and $\leq$ on vectors and matrices should be understood *elementwise* and $P \succ 0$ ($P \prec 0$) indicates that P is a positive (negative) definite matrix. The operators $\oplus$ and $\odot$ denote the Minkowski sum and the linear image operators, respectively. The asterisk $\star$ represents the symmetric term in a symmetric matrix. For a signal $x_k \in \mathbb{R}^n$, its L_∞ norm is defined as: $\|x\|_\infty = \sup_{k\geq 0} \|x_k\|$, where $\|x_k\|^2 = x_k^T x_k$.

The following definitions and properties are essential in this paper.

Definition 1. (Zhang et al., 2019) For a set $\mathbf{X} \subset \mathbb{R}^n$, its interval hull $\mathrm{Box}(\mathbf{X})$ is defined as the smallest interval vector containing it, which is denoted as follows:

$$\mathbf{X} \subseteq \mathrm{Box}(\mathbf{X}) = [\underline{x}, \overline{x}], \tag{1}$$

where $[\underline{x}, \overline{x}] = \{x : x \in \mathbf{X},\ \underline{x} \leq x \leq \overline{x}\}$ is an interval vector, $\underline{x}$ and $\overline{x}$ denote the maximum lower and minimum upper bounds of x.

Definition 2. (Combastel, 2005) An s-order zonotope $\mathcal{Z} \subset \mathbb{R}^n$ $(n \leq s)$ is a linear image of a hypercube $\mathbf{B}^s = [-1,\ +1]^s$ as follows:

$$\mathcal{Z} = p \oplus H\mathbf{B}^s = \{z = p + Hb, b \in \mathbf{B}^s\} \tag{2}$$

where s and n are the order and dimension number of $\mathcal{Z}$, $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$ and $H \in \mathbb{R}^{n\times s}$ is called the generation matrix of $\mathcal{Z}$, which defines the shape and volume of $\mathcal{Z}$. For simplicity, we also use $\mathcal{Z} = \langle p, H\rangle$ to denote a zonotope.

Property 1. (Scott et al., 2014) For zonotpes, the following properties hold:

$$\Gamma \odot \langle p, H\rangle = \langle \Gamma p, \Gamma H\rangle \tag{3}$$

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1 \quad H_2]\rangle \tag{4}$$

where $p, p_1, p_2 \in \mathbb{R}^n$ are known vectors, $H, H_1, H_2 \in \mathbb{R}^{n\times s}$ and $\Gamma \in \mathbb{R}^{l\times n}$ are determined matrices.

Property 2. (Combastel, 2005) For an s-order zonotope $\mathcal{Z} = \langle p, H\rangle \subset \mathbb{R}^n$, its interval hull $\mathrm{Box}(\mathcal{Z}) = [\underline{z}, \overline{z}]$ can be obtained by

$$\begin{cases} \overline{z}(i) = p(i) + \sum_{j=1}^{s} |H(i,j)|,\ i = 1, \cdots, n, \\ \underline{z}(i) = p(i) - \sum_{j=1}^{s} |H(i,j)|,\ i = 1, \cdots, n. \end{cases} \tag{5}$$

According to the Definitions 1 and 2, the interval hull of zonotope $\mathcal{Z} = \langle p, H\rangle$ can also be computed by

$$\mathcal{Z} = p \oplus H\mathbf{B}^s \subseteq \mathrm{Box}(\mathcal{Z}) = p \oplus Rs(H)\mathbf{B}^n, \tag{6}$$

where $Rs(H) \in \mathbb{R}^{n\times n}$ is a diagonal matrix as follows:

$$Rs(H) = \mathrm{diag}\left(\left[\sum_{j=1}^{s} |H(1,j)| \quad \cdots \quad \sum_{j=1}^{s} |H(n,j)|\right]\right).$$

Remark 1. *In the application of zonotopic techniques, the column number of the generator matrix will increase linearly without the reduction operator. However, the reduction operator proposed in Combastel (2005) can be used to bound a zonotope by a lower-dimensional one, which can be described by the following equation*

$$\mathcal{Z} = \langle p, H\rangle \subseteq \langle p, \Omega(H)\rangle$$

where $p \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n\times s}$ are the center and generator matrix of $\mathcal{Z}$. $\Omega(H) \in \mathbb{R}^{n\times m}$ denotes the complexity *reduction operator with* $m\,(n \le m \le s)$ *is the maximum number of columns of the generated matrix. The* $\Omega(H)$ *can be computed as follows:*

- *Reordering the columns of matrix* H *in decreasing Euclidean norm and denote the obtained matrix as* $\hat{H}$*.*

$$\hat{H} = [h_1 \ \cdots \ h_s], \| h_j \| \geq \| h_{j+1} \|, \ j = 1, \cdots s-1.$$

- *Replacing the last* $s-m+n$ *smallest column of* $\hat{H}$ *by a diagonal matrix* $Rs(H_<) \in \mathbb{R}^{n\times n}$ *since the zonotope generated by these columns can be enclosed in a box.*

If $s \le m$, *then* $\Omega(H) = H$,

Else $\Omega(H) = [H_> \quad Rs(H_<)] \in \mathbb{R}^{n\times m}$,
where $H_> = [h_1 \ \cdots h_{m-n}]$, $H_< = [h_{m-n+1} \ \cdots h_m]$.

3. PROBLEM FORMULATION

Consider the following discrete-time linear switched system subject to unknown disturbances and noises

$$\begin{cases} x_{k+1} = A_{\sigma(k)}x_k + B_{\sigma(k)}u_k + D_{\sigma(k)}w_k \\ y_k = C_{\sigma(k)}x_k + E_{\sigma(k)}v_k \end{cases}, \quad k \in \mathbb{R}_+, \quad (7)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ are the state vector, the input vector, the output vector, the unknown disturbances and the measurement noises, respectively. $\sigma(k)$ is a known piecewise constant function which denotes the switching signal. $\left\{\left(A_{\sigma(k)}, B_{\sigma(k)}, C_{\sigma(k)}, D_{\sigma(k)}, E_{\sigma(k)}\right) : \sigma(k) \in \mathscr{N}\right\}$ are a family of matrices parameterized by an index set $\mathscr{N} = \{1, \cdots, N\}$ and N is the number of subsystems. Let $q = \sigma(k)$ be the index of the active subsystem, A_q, B_q, C_q, D_q and E_q are known constant matrices with the corresponding dimensions.

The following assumptions will be used in this paper.

Assumptions 1. The switching signal $\sigma(k)$ in (7) can be available in real-time.

Assumptions 2. The initial system state vector x_0, the disturbances w_k and the noises v_k in (7) are assumed to be unknown but bounded as follows:

$$|x_0 - p_0| \le \tilde{x}_0, \ |w_k| \le \tilde{w}, \ |v_k| \le \tilde{v}, \quad (8)$$

where $|\cdot|$ denotes the absolute value operator, $p_0 \in \mathbb{R}^{n_x}$, $\tilde{x}_0 > 0 \in \mathbb{R}^{n_x}$, $\tilde{w} > 0 \in \mathbb{R}^{n_w}$ and $\tilde{v} > 0 \in \mathbb{R}^{n_v}$ are known vectors.

According to Definition 1, (8) can be reformulated as

$$x_0 \in \langle p_0, H_0\rangle, \ w_k \in \langle 0, W\rangle, \ v_k \in \langle 0, V\rangle, \quad (9)$$

where p_0 is a known vector, H_0, W and V are diagonal matrices with their diagonal elements equal to $\tilde{x}_0$, $\tilde{w}$ and $\tilde{v}$, respectively. For simplicity, we denote $x_0 \in \mathcal{X}_0 = \langle p_0, H_0\rangle$, $w_k \in \mathcal{W} = \langle 0, W\rangle$ and $v_k \in \mathcal{V} = \langle 0, V\rangle$.

The objective of the interval estimation method is to obtain an interval vector $[\underline{x}_k, \overline{x}_k]$, which can contain the real state x_k, i.e.

$$\underline{x}_k \le x_k \le \overline{x}_k, \quad k \in \mathbb{R}_+.$$

In this paper, we propose a novel interval estimation method for discrete-time linear switched systems by integrating the robust state observer design with the zonotopic analysis. First, a class of Luenberger observers for system (7) are designed via the L_∞ technique such that their estimation errors are robust against the unknown inputs. Based on the designed observers, the interval estimation will be determined with the aid of zonotopic analysis.

4. ROBUST STATE OBSERVER DESIGN

Consider the following observer for the system (7)

$$\hat{x}_{k+1} = A_q\hat{x}_k + B_q u_k + L_q(y_k - C_q\hat{x}_k), \quad (10)$$

where $\hat{x}_k$ is the state estimation vector and $L_q \in \mathbb{R}^{n_x\times n_y}$, $q \in \mathscr{N}$ is the observer gain matrix to be designed.

Define the estimation error as

$$e_k = x_k - \hat{x}_k, \quad (11)$$

then following error dynamics systems can be obtained

$$e_{k+1} = (A_q - L_qC_q)e_k + D_q w_k - L_qE_qv_k, \quad (12)$$

which can be rewritten as

$$e_{k+1} = A_{eq}e_k + D_qw_k + L_{eq}v_k, \quad (13)$$

where $A_{eq} = A_q - L_qC_q$ and $L_{eq} = -L_qE_q$, $q \in \mathscr{N}$.

In order to obtain accurate estimation, L_∞ technique is used to attenuate the effect of disturbances and noises on the estimation error. For error dynamic systems (12), the following theorem is proposed to design L_q for the observer in (10) such that the estimation error is robust against the unknown disturbances and noises.

Theorem 1. Given scalars $\gamma_w > 0$, $\gamma_v > 0$ and $0 < \lambda < 1$, if there exist a constant $\mu > 0$, matrices $P = P^T \succ 0 \in \mathbb{R}^{n_x\times n_x}$ and $W_q \in \mathbb{R}^{n_x\times n_y}$ for $\forall q \in \mathscr{N}$ such that

$$\begin{bmatrix} (\lambda-1)P & \star & \star & \star \\ 0 & -\mu I_{n_w} & \star & \star \\ 0 & 0 & -\mu I_{n_v} & \star \\ PA_q - W_qC_q & PD_q & -W_qE_q & -P \end{bmatrix} \prec 0, \quad (14)$$

$$\begin{bmatrix} \lambda P & \star & \star & \star \\ 0 & (\gamma_w-\mu)I_{n_w} & \star & \star \\ 0 & 0 & (\gamma_v-\mu)I_{n_v} & \star \\ I_{n_x} & 0 & 0 & (\gamma_w+\gamma_v)I_{n_x} \end{bmatrix} \succ 0, \quad (15)$$

then the error system in (12) is asymptotically stable and satisfies the following L_∞ performance

$$\|e_k\|^2 \le (\gamma_w+\gamma_v)(\lambda(1-\lambda)^kV_0 + \gamma_w\|w\|^2 + \gamma_v\|v\|^2), \quad (16)$$

where $V_0 = e_0^TPe_0$ and $P \succ 0 \in \mathbb{R}^{n_x\times n_x}$ being a designed matrix. Moreover, if the LMIs in (14) and (15) are solvable, the matrix L_q can be determined by $L_q = P^{-1}W_q$, $q \in \mathscr{N}$.

Proof : Choose the following common quadratic Lyapunov function

$$V_k = e_k^TPe_k, P = P^T \succ 0, \quad (17)$$

Then, the following equation can be obtained

$$\Delta V_k = V_{k+1} - V_k = \begin{bmatrix} e_k \\ w_k \\ v_k \end{bmatrix}^T \Phi \begin{bmatrix} e_k \\ w_k \\ v_k \end{bmatrix}, \quad (18)$$

where

$$\Phi = \begin{bmatrix} A_{eq}^TPA_{eq} - P & \star & \star \\ D_q^TPA_{eq} & D_q^TPD_q & \star \\ L_{eq}^TPA_{eq} & L_{eq}^TPD_q & L_{eq}^TPL_{eq} \end{bmatrix}.$$

By setting $W_q = PL_q$, $q \in \mathscr{N}$ then according to $A_{eq} = A_q - L_qC_q$ and $L_{eq} = -L_qE_q$, the inequality in (14) is equivalent to

$$\begin{bmatrix} (\lambda-1)P & \star & \star & \star \\ 0 & -\mu I_{n_w} & \star & \star \\ 0 & 0 & -\mu I_{n_v} & \star \\ PA_{eq} & PD_q & PL_{eq} & -P \end{bmatrix} \prec 0, \quad (19)$$

Pre- and post- multiplying (19) with

$$\begin{bmatrix} I_{n_x} & 0 & 0 & A_{eq}^T \\ 0 & I_{n_w} & 0 & D_q^T \\ 0 & 0 & I_{n_v} & L_{eq}^T \end{bmatrix}$$

and its transpose, respectively, we have

$$\Phi + \begin{bmatrix} \lambda P & \star & \star \\ 0 & -\mu I_{n_w} & \star \\ 0 & 0 & -\mu I_{n_v} \end{bmatrix} \prec 0. \quad (20)$$

By pre-multiplying and post-multiplying (19) with $[e_k^T \; w_k^T \; v_k^T]$ and its transpose, we can obtain

$$\Delta V_k < -\lambda V_k + \mu w_k^T w_k + \mu v_k^T v_k. \quad (21)$$

When $w_k = 0$ and $v_k = 0$, (21) implies that

$$\Delta V_k = V_{k+1} - V_k < -\lambda V_k < 0 \quad (22)$$

Thus, the error system in (12) is asymptotically stable.

Furthermore, inequality (21) is equivalent to

$$V_{k+1} < (1-\lambda)V_k + \mu(\|\tilde{w}\|^2 + \|\tilde{v}\|^2),$$

which implies that

$$\begin{aligned} V_k &< (1-\lambda)^k V_0 + \mu \sum_{\tau=0}^{k-1} (1-\lambda)^\tau (\|\tilde{w}\|^2 + \|\tilde{v}\|^2) \\ &\le (1-\lambda)^k V_0 + \mu \frac{1-\lambda^k}{\lambda} (\|\tilde{w}\|^2 + \|\tilde{v}\|^2) \\ &\le (1-\lambda)^k V_0 + \frac{\mu\|\tilde{w}\|^2}{\lambda} + \frac{\mu\|\tilde{v}\|^2}{\lambda}. \end{aligned} \quad (23)$$

By using the Schur complement lemma (Boyd et al., 1994), the inequality in (15) is equivalent to

$$\begin{bmatrix} \lambda P & \star & \star \\ 0 & (\gamma_w - \mu) I_{n_w} & \star \\ 0 & 0 & (\gamma_v - \mu) I_{n_v} \end{bmatrix} - \frac{1}{\gamma_w + \gamma_v} \begin{bmatrix} I_{n_x} \\ 0 \\ 0 \end{bmatrix} [I_{n_x} \; 0 \; 0] \succ 0, \quad (24)$$

Pre-multiplying and post-multiplying inequality (24) with $[e_k^T \; w_k^T \; v_k^T]$ and its transpose, we have

$$e_k^T e_k \le (\gamma_w + \gamma_v)(\lambda V_k + (\gamma_w - \mu)\|\tilde{w}\|^2 + (\gamma_v - \mu)\|\tilde{v}\|^2). \quad (25)$$

Substituting (23) into (25) yields

$$\begin{aligned} e_k^T e_k &\le (\gamma_w + \gamma_v)(\lambda((1-\lambda)^k V_0 + \frac{\mu\|\tilde{w}\|^2}{\lambda} + \frac{\mu\|\tilde{v}\|^2}{\lambda}) \\ &\quad + (\gamma_w - \mu)\|\tilde{w}\|^2 + (\gamma_v - \mu)\|\tilde{v}\|^2) \\ &= (\gamma_w + \gamma_v)(\lambda(1-\lambda)^k V_0 + \gamma_w \|\tilde{w}\|^2 + \gamma_v \|\tilde{v}\|^2), \end{aligned}$$

it follows that the L_∞ performance holds. □

Remark 2. *In order to attenuate the influence of disturbances and noises as much as possible, the minimal scalars γ_w and γ_v can be obtained by solving the following optimization problem:*

$$\min \quad \gamma_w + \gamma_v, \quad (26a)$$

$$\text{s.t.} \quad (14) - (15) \quad (26b)$$

and the feasible solution gives the observer gain matrix by $L_q = P^{-1} W_q$, $q \in \mathscr{N}$.

Remark 3. *For brevity, the robust observer in (10) is designed by a common Lyapunov function, which may result in some conservatism. In fact, the proposed observer can also be determined by using multiple Lyapunov functions, which may reduce such conservatism and further improve the estimation accuracy (Shi et al., 2015; Fei et al., 2017).*

5. INTERVAL ESTIMATION OF STATE

After getting observer gain matrices L_q, $q \in \mathscr{N}$ by solving the optimization problem (26), the interval estimation of x_k can be obtained based on the zonotopic techniques.

From (11), we can obtain

$$x_k = \hat{x}_k + e_k. \quad (27)$$

Consequently, if an interval vector $[\underline{e}_k, \overline{e}_k]$ satisfying $\underline{e}_k \le e_k \le \overline{e}_k$, $k \in \mathbb{R}_+$ can be obtained, from (27), the interval estimation of x_k can be calculated as follows:

$$\begin{cases} \overline{x}_k = \hat{x}_k + \overline{e}_k, \\ \underline{x}_k = \hat{x}_k + \underline{e}_k. \end{cases} \quad (28)$$

Therefore, the interval estimation of x_k can be transformed as interval analysis of the estimation error e_k. In the following, we will first obtain the interval estimation of e_k, and then give that of x_k.

Based on the zonotopic approach, the interval estimation of x_k can be realised by using the following theorem.

Theorem 2. For the observer (10) and the error dynamics systems (12), given $p_0 = \hat{x}_0$, then state x_k is bounded in a zonotope $\mathcal{X}_k = \langle \hat{x}_k, H_k \rangle$ and the interval estimation $[\underline{x}_k, \overline{x}_k]$ of x_k can be obtained as follows:

$$\begin{cases} \overline{x}_k(i) = \hat{x}_k(i) + \sum_{j=1}^{m} |H_k(i,j)|, \; i = 1, \cdots, n, \\ \underline{x}_k(i) = \hat{x}_k(i) - \sum_{j=1}^{m} |H_k(i,j)|, \; i = 1, \cdots, n, \end{cases} \quad (29)$$

where m is the column number of H_k and H_k satisfies the following iteration equation

$$H_{k+1} = [(A_q - L_q C_q)\Omega(H_k) \;\; D_q W \;\; -L_q E_q V]. \quad (30)$$

Proof : We first show the interval estimation of x_k can be obtained from (29). When $\mathcal{X}_0 = \langle \hat{x}_0, H_0 \rangle$, then from (3) and (11), we have

$$e_0 \in \mathcal{E}_0 = \langle \hat{x}_0, H_0 \rangle \oplus (-\hat{x}_0) = \langle 0, H_0 \rangle. \quad (31)$$

Note that $w_k \in \langle 0, W \rangle$, $v_k \in \langle 0, V \rangle$ and $e_0 \in \langle 0, H_0 \rangle$, thus we can conclude that $e_k \in \mathcal{E}_k = \langle 0, H_k \rangle$. From (27), we have $x_k \in \mathcal{X}_k = \hat{x}_k \oplus \langle 0, H_k \rangle = \langle \hat{x}_k, H_k \rangle$. Using Property 2, the interval estimation of x_k can be determined as

$$\begin{cases} \overline{x}_k(i) = \hat{x}_k(i) + \sum_{j=1}^{m} |H_k(i,j)|, \; i = 1, \cdots, n, \\ \underline{x}_k(i) = \hat{x}_k(i) - \sum_{j=1}^{m} |H_k(i,j)|, \; i = 1, \cdots, n. \end{cases}$$

We now prove the iteration equation in (30). Since $e_k \in \mathcal{E}_k = \langle 0, H_k \rangle$, then based on (9) and (12), $e_{k+1} \in \hat{\mathcal{E}}_{k+1}$ is updated as follows:

$$\begin{aligned} \hat{\mathcal{E}}_{k+1} &= \left\langle 0, \hat{H}_{k+1} \right\rangle \\ &= (A_q - L_q C_q) \odot \mathcal{E}_k \oplus D_q \odot \mathcal{W} \oplus (-L_q E_q) \odot \mathcal{V}. \end{aligned}$$

According to (3) and (4), $\hat{H}_{k+1}$ can be written as

$$\hat{H}_{k+1} = [(A_q - L_q C_q) H_k \;\; D_q W \;\; -L_q E_q V].$$

Using the reduction operator in Remark 1, we can obtain $\langle 0, H_k \rangle \subseteq \langle 0, \Omega(H_k) \rangle$, and it follows that $\langle 0, \hat{H}_{k+1} \rangle \subseteq \langle 0, H_{k+1} \rangle$. Finally, we have $e_{k+1} \in \mathcal{E}_{k+1} = \langle 0, H_{k+1} \rangle$. □

Remark 4. *It can be seen that the proposed method does not requires cooperative constraints and can avoid the additional conservatism caused by coordinate transformation.*

Therefore, the proposed approach provides a systematic way to improve the interval estimation accuracy by combining robust observer design and zonotopic techniques.

6. SIMULATION

In this section, a numerical example adapted from Dinh et al. (2019) is utilized to illustrate the feasibility and effectiveness of the proposed method. Consider the following discrete-time linear switched system with the unknown disturbances and measurement noises described by

$$\begin{cases} x_{k+1} = A_q x_k + B_q u_k + D_q w_k \\ y_k = C_q x_k + E_q v_k \end{cases}, \quad q = 1, \cdots, 3. \tag{32}$$

where

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0.3 & -2 \\ 0 & 0.6 \end{bmatrix}, A_3 = \begin{bmatrix} 0.5 & -1.1 \\ 0 & 0.16 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$C_1 = [0.2 \ \ 0.8], C_2 = [1 \ \ 0], C_3 = [0.1 \ \ 1],$$

$$D_1 = D_2 = D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = E_2 = E_3 = 1.$$

The switching signal $\sigma(k)$ between the three subsystems is plotted in Fig 1. By solving the optimization problem (26), we can obtain $\lambda = 0.5$, $\mu = 5.7532$, $\gamma_w = 5.7548$, $\gamma_v = 5.7543$, and the gain matrices L_1, L_2 and L_3 as

$$L_1 = \begin{bmatrix} -0.0953 \\ 0.1362 \end{bmatrix}, L_2 = \begin{bmatrix} 0.5519 \\ -0.0792 \end{bmatrix}, L_3 = \begin{bmatrix} -1.0969 \\ 0.1189 \end{bmatrix}.$$

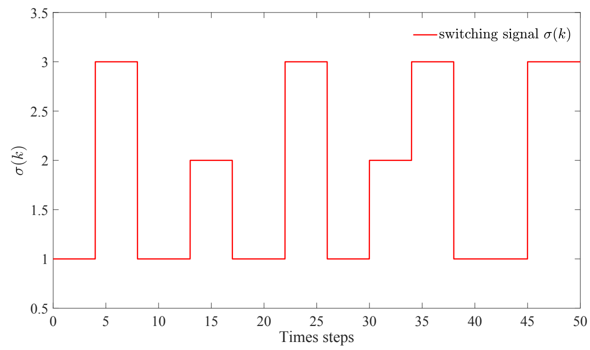

Fig. 1. Switching signal $\sigma(k)$

In the simulation, we set the input $u_k = 0.5\sin(0.1k)$ and the initial state vector $x_0 = [1 \ \ 2]^T$. The unknown disturbances and measurement noises are set as $w_k = 0.1[\sin(0.5k) \ \ \cos(0.5k)]^T$ and $v_k = 0.1\sin(0.5k)$. The initial zonotope of x_0 are set as $p_0 = [1 \ \ 1]^T$ and $H_0 = I_2$. The generation matrices of $\mathcal{W}$ and $\mathcal{V}$ are set as $W = 0.1I_2$ and $V = 0.1I_1$. Meanwhile, we set the reduction order of the matrix $\Omega(H_k)$ as $m = 20$ to limit the column number of the generator matrix.

The simulation results are shown in Fig 2. As shown in Fig 2, the proposed method is able to decrease the effect of the unknown disturbances and measurement noises. Although there is initial estimation error, the states estimate can quickly track the states and give more accurate interval estimation results. In the simulation study, the proposed method is compared with the optimal interval observers proposed in Dinh et al. (2019). Note that the optimal interval observers proposed in Dinh et al. (2019) have not

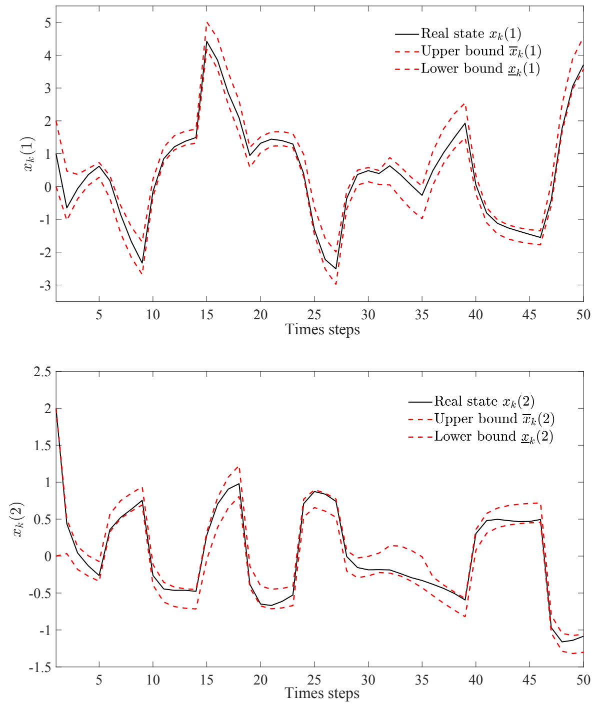

Fig. 2. States and their interval estimations by the proposed method

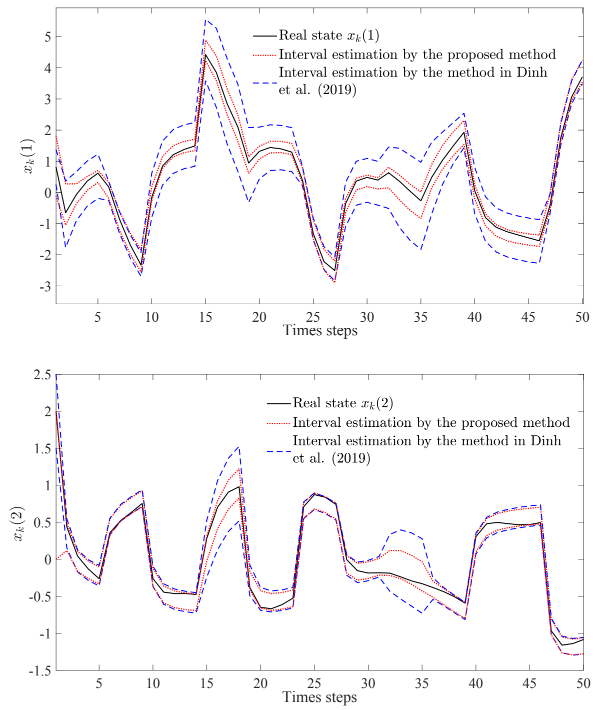

Fig. 3. States and their interval estimations by the proposed method and by the method in Dinh et al. (2019)

considered the influence of unknown measurement noises. Thus, we set $E_1 = E_2 = E_3 = 0$ and $v_k = 0$ of system (32). By solving (26), we have $\lambda = 0.5$, $\mu = 7.5644$, $\gamma_w = 7.5654$, and the observer gain matrices L_1, L_2 and L_3 can be determined as follows:

$$L_1 = \begin{bmatrix} -0.2333 \\ 0.1902 \end{bmatrix}, L_2 = \begin{bmatrix} 0.6087 \\ -0.0929 \end{bmatrix}, L_3 = \begin{bmatrix} -1.4913 \\ 0.1845 \end{bmatrix}.$$

Manuscript 1087 submitted to 21st IFAC World Congress, 2020.
Received November 7, 2019.

The simulation results are shown in Fig 3. The results show that the interval estimation obtained by the proposed method is more accurate than optimal interval observers. Therefore, the results show the feasibility and effectiveness of our method in interval estimation of state.

7. CONCLUSIONS

This paper studies interval estimation for discrete-time linear switched systems affected by bounded disturbances and noises. A novel interval estimation method is proposed via the robust observer design and zonotopic techniques. Compared with interval observers, the proposed method overcomes the cooperativity constraints and avoids the additional conservatism caused by coordinate transformation. Finally, numerical simulations have demonstrated the feasibility and effectiveness of the proposed interval estimation approach. In the future, we will focus on using the multiple Lyapunov functions to further improve the estimation performance of the proposed method and this will be our next research work.

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