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# Optimal interval observer for switched Takagi-Sugeno systems: an application to interval fault estimation

Yosr Garbouj<sup>#</sup>, Thach N. Dinh<sup>#,\*</sup>, Tarek Raïssi, Talel Zouari and Moufida Ksouri

**Abstract**—The main goal of this paper is to design interval observers for continuous nonlinear switched systems. The nonlinear modes are described by the multimodel approach of Takagi-Sugeno (T-S) fuzzy systems where premise variables depending on the state vector which is unmeasurable. In this paper, we propose T-S interval observers that consider the unmeasurable premise variables as bounded uncertainties under common assumptions that additive disturbances as well as measurement noises are unknown but bounded. The stability and the nonnegativity conditions are given in terms of Linear Matrix Inequality (LMI) to ensure simultaneously the convergence and the nonnegativity of error dynamics. Furthermore, in the absence of measurement noises, optimal gains attenuating the effect of additive disturbances are computed using  $H_\infty$  approach to improve the accuracy of the present interval observers. Theoretical results are finally applied to a numerical example to highlight the performance of the proposed method.

**Index Terms**—Nonlinear switched systems, T-S interval observer, unmeasurable premise variable, stability,  $H_\infty$  formalism.

## I. INTRODUCTION

MANY engineering applications evolve through the coupling between continuous and discrete dynamics where a collection of indexed modes interacts with a switching signal that selects the active one at each time instant. Such systems are called *hybrid systems* which have attracted an ever growing attention [1], [2]. Meanwhile, *switched systems* are an important class of hybrid systems [3] and many great efforts have been paid to the studies of this class due to its appearance in a large number of physical applications such as biological systems [4], robotic systems [5], embedded systems [6], etc. Among these studies, some interesting results have been reported to deal with the problems of robust control [7], diagnosis [8] while others are particularly devoted to state estimation of such systems. More concretely, the case of linear switched systems are usually inspected [9]–[12]. However, the case of *nonlinear switched systems* has not been fully studied yet. Although in practice it is well-known that various domains

have to be described by nonlinear switched behaviors such as automotive [13], converters [14], network control applications [15] and so on.

Since several decades, *Takagi-Sugeno (T-S) fuzzy systems* have been considered as a powerful tool to cope with nonlinearities [16]. This approach is useful to analyze nonlinear systems by providing an effective representation of them [17]. T-S fuzzy models are based on fuzzy sets applied to a set of linear local models decomposing a nonlinear system into different zones. The validity of each one is quantified by a nonlinear weighting function depending on the so-called premise variables (or decision variables). Two main cases can be considered for premise variables: some works are devoted to the premise variables which are measurable (as the input or the output of the system) [18] while most of the literature deals with the second case for which the premise variables are composed of a subset of the system state  $x(t)$  [19], [20]. As widely known the former is conservative. Unlike the former, the latter is very challenging and appears in most of nonlinear mechanical systems. e.g., the readers can for instance refer to [21] for an example of a simple nonlinear mass-spring-damper mechanical system. Since the premise variables are endogenous and unknown, the T-S model defines a class of nonlinear systems with unknown nonlinearities which often hamper and sometimes prohibit construction of observers.

Following the remarkable development of fuzzy control systems, nonlinear switched plants can be also described in the compact form of T-S fuzzy models. In the literature, few works are devoted to this family of systems [18], [22], [23]. However, the previously mentioned works adopt a restrictive requirement of the premise variables: *they are supposed to be measurable*.

Recently, designing observers for such switching representations with unmeasurable premise variables becomes an open issue to tackle. In addition, the case of systems subject to disturbances and measurement noises is more challenging as it is the case for most of real-life systems. However, this case has not been fully investigated in the literature. Conventional observer may not be an efficient method to deal with uncertainties. Additionally, for the purpose of control such as stabilization and tracking, precise information of the state vector in transient periods is not necessary. However, practically there is a great demand for estimation of the state of a system with guarantees at all time and the notion of *interval observer* has been one of useful approaches to meeting this practical demand. That is why interval observers have

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been fertile ground for studies in the past few years. In the literature, interval observers have been successfully applied to many physical systems such as biotechnological processes [24], biochemical processes [25], energy generation system [26]. It has been shown in many contributions that interval observers do not only give interval estimates but also can be employed for stabilization by feedback control [27]–[29]. Although design of interval observers requires bounds of the uncertainties/disturbances and bounds of the initial conditions to be known a priori, this requirement is accepted in many applications. Interval observer designs for several classes of systems have been reviewed in both continuous and discrete time, e.g., linear systems [30]–[33], bilinear systems [28], nonlinear systems [34]–[36], time-delay systems [37], linear switched systems [38]–[41].

According to the above-mentioned studies, few results of interval observer designs are devoted to nonlinear switched systems represented by T-S fuzzy models. It is worth noting that most of existing works in the literature concerning interval estimator design for fuzzy systems, even in the non-switching case, have coped only with the *measurable* premise variables [26], [42]. To the best of the authors' knowledge, designing interval observers with the constraints of *unmeasurable* premise variables has not been fully considered yet. This motivates the present work. The goal of this paper is to propose new results of T-S interval observer designs for nonlinear switched systems subject to measurement noises and additive disturbances. Using the center-of-gravity method for defuzzification, nonlinear switched systems can be represented by T-S fuzzy models with unmeasurable premise variables. The first contribution is to construct interval observers for this class of fuzzy systems. The stability and the nonnegativity properties are both given in terms of Linear Matrix Inequalities (LMIs). The former is achieved via a common Lyapunov function while the latter is inspired by a simple but interesting fact introduced in [43] (see Lemma 3). The second contribution of this paper is to deal with the problem of the interval width optimization in the absence of measurement noises. To the best of the authors' knowledge, this optimization issue has not been fully treated. Only the case of discrete-time linear systems has been considered in [44]. The idea is based on *H<sub>∞</sub> approach* to compute optimal gains which attenuate the effect of the system's disturbances on the estimation error. Thus, the optimal gains ensure a tighter interval width which improves the accuracy of the interval between the lower and upper bounds.

Compared to previous works, the contributions of this paper are summarized as follows:

- 1) The proposed scheme is applicable to T-S model with premise variables composed of a subset of the system state. Note that most of the designs available in the literature are carried out for cooperative systems (sometimes after a coordinate transformation) but the key advantage of the new construction is the simplicity of its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system. It is the first time such design is introduced.

- 2) For a given lower and upper bounds of disturbances and in the absence of measurement noises, an optimal interval estimation for the state can be obtained based on an energy-bounded design method. To the best of our knowledge, this is the first interval observer that can attenuate the effect of disturbances for T-S systems with unmeasurable premise variables.
- 3) An application to sensor fault detection is proposed. Simulations are given to highlight the effectiveness of the proposed schemes.

The paper has the following structure: preliminaries and the system description are given in Section II. Section III is devoted to the main results: (i) designing T-S interval observers, (ii) based on *H<sub>∞</sub> approach*, optimizing the interval width by computing optimal gains and (iii) a fault detection scheme is introduced as an application. Two examples are considered in Section IV: the first one draws comparative simulations while the second one illustrates the efficiency of the proposed method in the sensor fault detection. Section V concludes the paper.

## II. PRELIMINARIES AND SYSTEM DESCRIPTION

### A. Notions, definitions and lemmas

The set of real numbers is denoted by  $\mathbb{R}$  and the set of nonnegative real numbers is denoted by  $\mathbb{R}_{\geq 0}$ , i.e.,  $\mathbb{R}_{\geq 0} := [0, +\infty)$  where  $\mathbb{R}_{\geq 0}^{n \times n}$  is of dimension  $n \times n$ . Inequalities are understood *component-wise*, i.e., for  $x_a = [x_{a,1}, \dots, x_{a,n}]^T \in \mathbb{R}^n$  and  $x_b = [x_{b,1}, \dots, x_{b,n}]^T \in \mathbb{R}^n$ ,  $x_a \leq x_b$  if and only if, for all  $i \in \{1, \dots, n\}$ ,  $x_{a,i} \leq x_{b,i}$ .  $I$ ,  $0$  denote respectively identity and zero matrices with appropriate dimensions. A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be positive definite and written as  $\alpha \in \mathcal{P}$  if it is continuous and satisfies  $\alpha(0) = 0$  and  $\alpha(s) > 0$  for all  $s \in (0, \infty)$ . A function  $\alpha \in \mathcal{P}$  is said to be of class  $\mathcal{K}$  if it is strictly increasing. If in addition  $\alpha$  is unbounded, then it is of class  $\mathcal{K}_{\infty}$ . The symbol  $P \succ 0$  (resp.  $P \prec 0$ ) means that the symmetric matrix  $P$  is positive (resp. negative) definite.  $E_p$  is a  $(p \times 1)$  vector whose elements are equal to 1.  $I_n$  is the identity matrix with dimension  $n \times n$ . For  $x(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ , the  $\mathcal{L}_2$  norm is denoted by  $\|x(t)\|_2$ . For a measurable and locally essentially bounded input  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ , the symbol  $\|u\|_{[t_0, t_1]}$  denotes its  $\mathcal{L}_{\infty}$  norm. If  $t_1 = +\infty$ , then we will simply write  $\|u\|$ . The  $\mathcal{L}_{\infty}$  is denoted as the set of all inputs  $u$  such that  $\|u\| < \infty$ . The left and right endpoints of an interval  $[x(t)]$  are denoted by  $\underline{x}(t)$  and  $\bar{x}(t)$  such as  $[x(t)] = [\underline{x}(t), \bar{x}(t)]$ . A matrix  $A \in \mathbb{R}^{n \times n}$  is called Metzler if all the off-diagonal elements are nonnegative. A matrix  $A \in \mathbb{R}^{n \times n}$  is said to be nonnegative if each entry of  $A$  is nonnegative. Given a matrix  $A \in \mathbb{R}^{m \times n}$ , we define  $A^+ = \max\{0, A\}$ ,  $A^- = A^+ - A$  and denote the absolute value of a matrix by  $|A| = A^+ + A^-$  (similarly for vectors). For square matrices

$$T_i, \text{ we define } \text{diag}([T_1 \dots T_N]) = \begin{bmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & T_N \end{bmatrix}.$$

*Lemma 1:* [45] The system described by:

$$\dot{x}(t) = Ax(t) + u(t) \quad (1)$$

is said to be nonnegative if  $A$  is a Metzler matrix and  $u(t) \geq 0$ . For any initial condition  $x(0) \geq 0$ , the solution of (1) satisfies  $x(t) \geq 0, \forall t \geq 0$ .

**Lemma 2:** [46] Let  $x \in \mathbb{R}^n$  be a vector such that  $\underline{x} \leq x \leq \bar{x}$ . (1) if  $A \in \mathbb{R}^{m \times n}$  is a constant matrix, then

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x} \quad (2)$$

(2) if  $A \in \mathbb{R}^{m \times n}$  is a matrix satisfying  $\underline{A} \leq A \leq \bar{A}$ , for some  $\underline{A}, \bar{A} \in \mathbb{R}^{m \times n}$ , then

$$\begin{aligned} \underline{A}^+ \underline{x}^+ - \bar{A}^+ \underline{x}^- - \underline{A}^- \bar{x}^+ + \bar{A}^- \bar{x}^- &\leq Ax \\ &\leq \bar{A}^+ \bar{x}^+ - \underline{A}^+ \bar{x}^- - \bar{A}^- \underline{x}^+ + \underline{A}^- \underline{x}^- \end{aligned} \quad (3)$$

**Lemma 3:** [43] A matrix  $A \in \mathbb{R}^{n \times n}$  is Metzler if and only if there exists  $\eta \in \mathbb{R}_{\geq 0}$  such that  $A + \eta I \in \mathbb{R}_{\geq 0}^{n \times n}$ .

Consequently, if there exist a positive diagonal matrix  $P \in \mathbb{R}^{n \times n}$  and a constant  $\eta > 0$  such that

$$PA + \eta P \geq 0, \quad (4)$$

then,  $A$  is Metzler.

**Lemma 4:** [47] Consider  $x$  and  $y$  with appropriate dimensions and  $\Omega$  a positive definite matrix. the following property is verified:

$$x^T y + y^T x \leq x^T \Omega x + y^T \Omega^{-1} y \quad (5)$$

**Lemma 5:** [48] Let  $\lambda > 0$  be a scalar and  $P \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix, then:

$$2x^T y \leq \frac{1}{\lambda} x^T P x + \lambda y^T P^{-1} y \quad x, y \in \mathbb{R}^n \quad (6)$$

**Lemma 6 (Input-to-State Stability (ISS) [49]):** Consider the linear switched system with inputs

$$\dot{x}(t) = A^\sigma x(t) + d(t), \quad \sigma \in \{1, 2, \dots, N\} \quad (7)$$

Suppose that there exist a continuously differentiable positive definite, radially unbounded function  $V : \mathbb{R}^n \rightarrow [0, \infty)$ , a class  $\mathcal{K}_\infty$  function  $\alpha$ , a class  $\mathcal{K}$  function  $\gamma$  and a number  $\varphi > 0$  such that

$$\dot{V} \leq -\varphi \alpha(V) + \gamma(\|d\|) \quad (8)$$

along all trajectories of (7). Then, the linear switched system (7) is Input-to-State Stable (ISS). The ISS is used to study the stability of a system with external inputs. The system is ISS if it is globally asymptotically stable in the absence of external inputs and if its trajectories are bounded by a function of the size of the input for all sufficiently large times.

**Definition 1 (Interval observer for switched systems [50]):** Consider a switched system:

$$\begin{cases} x(k+1) = f_q(x(k), d(k)), \\ y(k) = g_q(x(k)), \end{cases} \quad (9)$$

with the state  $x \in \mathbb{R}^n$ , the output  $y \in \mathbb{R}^p$ , the index of the active subsystem  $q \in \overline{1, N}$ , the number of subsystems is  $N \in \mathbb{N}$  and  $f_q, g_q$  are functions. The uncertainties  $d(k) \in \mathbb{R}^\ell$  are such that there exists a sequence  $\bar{d}(k) \in \mathbb{R}^\ell$  where, for all  $k \geq 0$ ,  $-\bar{d}(k) \leq d(k) \leq \bar{d}(k)$ . The initial condition  $x(0)$ , is assumed to be bounded by two known bounds:

$$\underline{x}(0) \leq x(0) \leq \bar{x}(0). \quad (10)$$

Then, the dynamical system

$$z(k+1) = h_q(z(k), y(k), \bar{d}(k)), \quad q \in \overline{1, N}, \quad N \in \mathbb{N}, \quad (11)$$

associated with the initial condition  $z(0) = r_q(\bar{x}(0), \underline{x}(0)) \in \mathbb{R}^{n_z}$  and bounds for the solution  $x(k)$ :  $\bar{x}(k) = \bar{h}_q(z(k), y(k))$ ,  $\underline{x}(k) = \underline{h}_q(z(k), y(k))$ , where  $q \in \overline{1, N}$ ,  $N \in \mathbb{N}$ ,  $h_q, r_q, \bar{h}_q$  and  $\underline{h}_q$  are functions, is called (i) a **framer** for (9) if for any vectors  $x(0), \underline{x}(0)$  and  $\bar{x}(0)$  in  $\mathbb{R}^n$  satisfying (10), the solutions denoted respectively  $x$  and  $z$  of (9)-(11) with respectively  $x(0), z(0) = r_q(\bar{x}(0), \underline{x}(0))$  as initial condition at 0, satisfy for all  $k \geq 0$ , the inequalities

$$\underline{x}(k) = \underline{h}_q(z(k), y(k)) \leq x(k) \leq \bar{h}_q(z(k), y(k)) = \bar{x}(k), \quad (12)$$

(ii) an **interval observer** for (9) if in addition  $|\bar{h}_q(z(k), y(k)) - \underline{h}_q(z(k), y(k))|$  is input-to-state stable (ISS) with respect to  $d(k) \in \mathbb{R}^\ell$  for all  $q \in \overline{1, N}$ ,  $N \in \mathbb{N}$ .

## B. Uncertain T-S model formulation for nonlinear switched systems

Consider a continuous time nonlinear switched system described as follows :

$$\begin{aligned} \Sigma_{\sigma(t)} : \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t), d(t)) \\ y(t) = g_{\sigma(t)}(x(t), v(t)) \end{cases} \quad (13) \\ \sigma(t) : \mathbb{R}_{\geq 0} \rightarrow \{1, 2, \dots, N\} \end{aligned}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input and  $y(t) \in \mathbb{R}^p$  is the output.  $d(t) \in \mathbb{R}^n$  and  $v(t) \in \mathbb{R}^p$  are respectively the bounded additive disturbances and measurement noises.  $\sigma(t)$  is the switching law such that  $\sigma(t) \in \{1, \dots, N\}$  is the index of the active mode. For example, if one has  $\sigma(t) = i$ ,  $i \in \{1, 2, \dots, N\}$ , the system is said to be in the mode  $i$  at the instant  $t$ . For the sake of simplicity,  $\sigma(t)$  will be simply replaced by  $\sigma$ .  $f_\sigma$  and  $g_\sigma$  are nonlinear functions. The additive disturbances  $d(t)$  and the measurement noises  $v(t)$  are assumed to be unknown but bounded.

The first step is to approximate each nonlinear mode  $\sigma$  of the system (13) by T-S fuzzy models which consist of a set of linear sub-models interpolated through a weighting function  $\mu_i^\sigma(\xi(t))$  to contribute to the global behavior of the nonlinear switched system. Each linear model represents a local dynamics of the whole system. By using the center-of-gravity method [51], for defuzzification of the possible sets, the T-S representation is given by the following compact form:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(\xi(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \\ \forall \sigma \in \{1, 2, \dots, N\}, \forall i \in \{1, \dots, r\}, \end{cases} \quad (14)$$

where  $A_i^\sigma \in \mathbb{R}^{n \times n}$ ,  $B_i^\sigma \in \mathbb{R}^{n \times m}$  and  $C^\sigma \in \mathbb{R}^{p \times n}$  are known constant matrices.  $\mu_i^\sigma(\xi(t))$  are the weighting functions depending on the so-called decision variable  $\xi(t)$  that can be internal or external to the system. When these variables are internal, they can be measurable such as the input or the output of the system (i.e.  $\{u(t), y(t)\}$ ) or unmeasurable as the state

of the system (ie.  $x(t)$ ). The weighting functions verify the following convex sum properties:

$$\begin{cases} 0 \leq \mu_i^\sigma(\xi(t)) \leq 1, \forall \sigma \in \{1, \dots, N\}, \forall i \in \{1, \dots, r\} \\ \sum_{i=1}^r \mu_i^\sigma(\xi(t)) = 1 \end{cases} \quad (15)$$

*Assumption 1:*

$$\underline{d} \leq d(t) \leq \bar{d}, \quad |v(t)| \leq \bar{V}E_p, \quad \forall t \geq 0 \quad (16)$$

where  $\underline{d} = -\bar{d} \in \mathbb{R}^n$  and the scalar  $\bar{V}$  are a priori known.

Assumption 1 is basic in the literature of interval observers where the uncertainties are assumed bounded with known bounds.

*Assumption 2:* The state of the system  $x(t)$  and the known input vector  $u(t)$  are supposed to be bounded in norm.

In this work, we suppose that the weighting functions depend on the system state which is unmeasurable (i.e.  $\xi(t) = x(t)$ ). It is worth pointing out that considering the unknown state as a decision variable will improve the synchronization process of control or diagnosis of the system which is an important advantage of this system's structure. Therefore, the system (14) can be rewritten as:

$$\begin{cases} \begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(x(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{cases} \\ \forall \sigma \in \{1, 2, \dots, N\}, \forall i \in \{1, \dots, r\}. \end{cases} \quad (17)$$

Given the lower and upper bounds  $\underline{x}(t), \bar{x}(t), \in \mathbb{R}^n$  of the state  $x(t)$ . Then, inspired by the work [52], by adding and subtracting at the same time, respectively, the  $\sum_{i=1}^r \mu_i^\sigma(\bar{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t))$  term or the  $\sum_{i=1}^r \mu_i^\sigma(\underline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t))$  term and after rearranging, the second step consists in rewriting the system (17) with unmeasurable premise variables to an uncertain system with upper and lower estimated premise variables. The state vector is given in two equivalent ways as follows  $\forall \sigma \in \{1, 2, \dots, N\}$  and  $\forall i \in \{1, \dots, r\}$ :

$$\begin{cases} \begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) \\ \quad - \sum_{i=1}^r \bar{\delta}_i^\sigma(t)(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \end{cases} \\ \text{or} \\ \begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) \\ \quad + \sum_{i=1}^r \underline{\delta}_i^\sigma(t)(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \end{cases} \\ y(t) = C^\sigma x(t) + v(t) \end{cases} \quad (18)$$

with

$$\begin{aligned} \bar{\delta}_i^\sigma(t) &= \mu_i^\sigma(\bar{x}(t)) - \mu_i^\sigma(x(t)) \\ \underline{\delta}_i^\sigma(t) &= \mu_i^\sigma(x(t)) - \mu_i^\sigma(\underline{x}(t)) \end{aligned}$$

Thus, using  $\sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) = \sum_{i=1}^r \mu_i^\sigma(x(t)) = 1$  for all  $\underline{x}(t) \in \mathbb{R}^n$  and  $\bar{x}(t) \in \mathbb{R}^n$ , we obtain:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t))[(A_i^\sigma - \overline{\Delta A}^\sigma(t))x(t) + (B_i^\sigma - \overline{\Delta B}^\sigma(t))u(t)] + d(t) \\ \text{or} \\ \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t))[(A_i^\sigma + \underline{\Delta A}^\sigma(t))x(t) + (B_i^\sigma + \underline{\Delta B}^\sigma(t))u(t)] + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{cases} \quad (19)$$

where the uncertainties  $\overline{\Delta A}^\sigma(t)$ ,  $\overline{\Delta B}^\sigma(t)$  or  $\underline{\Delta A}^\sigma(t)$ ,  $\underline{\Delta B}^\sigma(t)$  are given by:

$$\begin{aligned} \overline{\Delta A}^\sigma(t) &= \sum_{i=1}^r \bar{\delta}_i^\sigma(t)A_i^\sigma, & \underline{\Delta A}^\sigma(t) &= \sum_{i=1}^r \underline{\delta}_i^\sigma(t)A_i^\sigma \\ &= \mathcal{A}^\sigma \bar{\Sigma}_A^\sigma(t)E_A & &= \mathcal{A}^\sigma \underline{\Sigma}_A^\sigma(t)E_A \end{aligned} \quad (20)$$

$$\mathcal{A}^\sigma = [A_1^\sigma \quad \dots \quad A_r^\sigma], \quad E_A = \begin{bmatrix} I_n & \dots & I_n \\ \underbrace{\hspace{10em}}_{r \text{ terms}} \end{bmatrix}^T \quad (21)$$

$$\bar{\Sigma}_A^\sigma(t) = \text{diag}([\bar{\delta}_1^\sigma(t)I_n \dots \bar{\delta}_r^\sigma(t)I_n]) \quad (22)$$

$$\underline{\Sigma}_A^\sigma(t) = \text{diag}([\underline{\delta}_1^\sigma(t)I_n \dots \underline{\delta}_r^\sigma(t)I_n]) \quad (23)$$

$$\begin{aligned} \overline{\Delta B}^\sigma(t) &= \sum_{i=1}^r \bar{\delta}_i^\sigma(t)B_i^\sigma, & \underline{\Delta B}^\sigma(t) &= \sum_{i=1}^r \underline{\delta}_i^\sigma(t)B_i^\sigma \\ &= \mathcal{B}^\sigma \bar{\Sigma}_B^\sigma(t)E_B & &= \mathcal{B}^\sigma \underline{\Sigma}_B^\sigma(t)E_B \end{aligned} \quad (24)$$

$$\mathcal{B}^\sigma = [B_1^\sigma \quad \dots \quad B_r^\sigma], \quad E_B = \begin{bmatrix} I_m & \dots & I_m \\ \underbrace{\hspace{10em}}_{r \text{ terms}} \end{bmatrix}^T \quad (25)$$

$$\bar{\Sigma}_B^\sigma(t) = \text{diag}([\bar{\delta}_1^\sigma(t)I_m \dots \bar{\delta}_r^\sigma(t)I_m]) \quad (26)$$

$$\underline{\Sigma}_B^\sigma(t) = \text{diag}([\underline{\delta}_1^\sigma(t)I_m \dots \underline{\delta}_r^\sigma(t)I_m]) \quad (27)$$

*Remark 1:* Due to the convex property of the weighting functions (15), we have  $-1 \leq \bar{\delta}_i^\sigma(t) \leq 1$  and  $-1 \leq \underline{\delta}_i^\sigma(t) \leq 1$ . The terms  $\bar{\Sigma}_A^\sigma(t)$ ,  $\underline{\Sigma}_A^\sigma(t)$ ,  $\bar{\Sigma}_B^\sigma(t)$  and  $\underline{\Sigma}_B^\sigma(t)$  satisfy  $\bar{\Sigma}_A^{\sigma T}(t)\bar{\Sigma}_A^\sigma(t) \leq I_{nr}$ ,  $\underline{\Sigma}_A^{\sigma T}(t)\underline{\Sigma}_A^\sigma(t) \leq I_{nr}$ ,  $\bar{\Sigma}_B^{\sigma T}(t)\bar{\Sigma}_B^\sigma(t) \leq I_{mr}$  and,  $\underline{\Sigma}_B^{\sigma T}(t)\underline{\Sigma}_B^\sigma(t) \leq I_{mr}$ .

*Assumption 3:* There exist known constant matrices  $\overline{\Delta A}_{\min}^\sigma$ ,  $\overline{\Delta A}_{\max}^\sigma$ ,  $\underline{\Delta A}_{\min}^\sigma$  and  $\underline{\Delta A}_{\max}^\sigma$  such that, for all  $t \geq 0$ , for all  $\sigma \in \{1, 2, \dots, N\}$ ,

$$\begin{aligned} \overline{\Delta A}_{\min}^\sigma &\leq \overline{\Delta A}^\sigma(t) \leq \overline{\Delta A}_{\max}^\sigma \\ \underline{\Delta A}_{\min}^\sigma &\leq \underline{\Delta A}^\sigma(t) \leq \underline{\Delta A}_{\max}^\sigma \end{aligned}$$

Similarly, there exist known constant matrices  $\overline{\Delta B}_{\min}^\sigma$ ,  $\overline{\Delta B}_{\max}^\sigma$ ,  $\underline{\Delta B}_{\min}^\sigma$  and  $\underline{\Delta B}_{\max}^\sigma$  such that, for all  $t \geq 0$ , for all  $\sigma \in \{1, 2, \dots, N\}$ ,

$$\begin{aligned} \overline{\Delta B}_{\min}^\sigma &\leq \overline{\Delta B}^\sigma(t) \leq \overline{\Delta B}_{\max}^\sigma \\ \underline{\Delta B}_{\min}^\sigma &\leq \underline{\Delta B}^\sigma(t) \leq \underline{\Delta B}_{\max}^\sigma \end{aligned}$$

Assumption 3 means that the uncertainties  $\overline{\Delta A}^\sigma(t)$ ,  $\overline{\Delta B}^\sigma(t)$ ,  $\underline{\Delta A}^\sigma(t)$  and  $\underline{\Delta B}^\sigma(t)$  are unknown terms but bounded by known constant matrices. This assumption is natural when dealing with system parameter uncertainties.

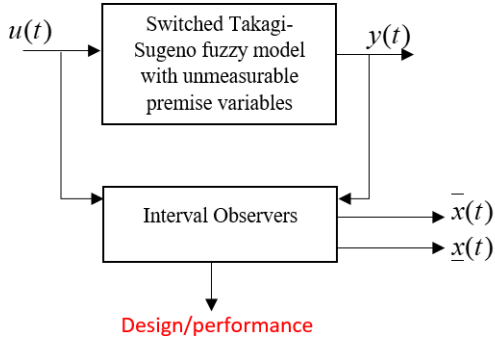


Fig. 1. A block diagram of the state estimation of the system (19) based on interval observer approach.

*Remark 2:* According to the design of the upper (respectively, lower) bound, first the (respectively, second) form of (19) is employed. The key idea is based on replacing unmeasurable premise variables  $\mu_i^\sigma(x(t))$  by their upper or lower estimated bounds (i.e.,  $\mu_i^\sigma(\bar{x}(t))$  or  $\mu_i^\sigma(\underline{x}(t))$ ) associating with respective bounded uncertainties  $(\overline{\Delta A}^\sigma(t), \overline{\Delta B}^\sigma(t))$  or  $(\underline{\Delta A}^\sigma(t), \underline{\Delta B}^\sigma(t))$ .

### III. MAIN RESULTS

In this paper, T-S interval observers are designed for a class of nonlinear switched systems described by fuzzy model-based approaches. The T-S fuzzy model can be represented in the compact form (17) with unmeasurable premise variables. Next, it is transformed into an uncertain switched T-S system (19) with upper or lower estimated premise variables. Starting from the initial state  $x(0)$  which verifies  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$  and taking into account the uncertainties, an interval observer will first be *designed*. The purpose is to satisfy stability and nonnegativity properties of the observation errors. Subsequently, gains of the present interval observer will be optimized using  $H_\infty$  approach to guarantee the attenuation of additive disturbances effect in the absence of measurement noises. Hence, these gains improve the estimation accuracy by tightening the error between upper and lower bounds. Thus, the *performance* of the interval observer will be improved. The main goal of this paper is illustrated in Figure 1.

#### A. T-S interval observer design

The third step of this paper is to design a T-S interval observer for the switched T-S system (19). Consider the following upper and lower dynamics with  $\sigma \in \{1, 2, \dots, N\}$  and  $i \in \{1, \dots, r\}$ :

$$\begin{cases} \dot{\bar{x}}(t) = \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) [(A_i^\sigma - L_i^\sigma C^\sigma) \bar{x}(t) + B_i^\sigma u(t) + \\ L_i^\sigma y(t) + |L_i^\sigma| \bar{V} E_p] + \bar{d} - \bar{\varphi}_{A,\min}^\sigma(t) - \bar{\varphi}_{B,\min}^\sigma(t) \\ \dot{\underline{x}}(t) = \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) [(A_i^\sigma - L_i^\sigma C^\sigma) \underline{x}(t) + B_i^\sigma u(t) + \\ L_i^\sigma y(t) - |L_i^\sigma| \bar{V} E_p] + \underline{d} + \underline{\varphi}_{A,\min}^\sigma(t) + \underline{\varphi}_{B,\min}^\sigma(t) \end{cases} \quad (28)$$

where

$$\begin{aligned} \bar{\varphi}_{A,\min}^\sigma(t) &= \overline{\Delta A}_{\min}^{\sigma+} \underline{x}^+(t) - \overline{\Delta A}_{\max}^{\sigma+} \underline{x}^-(t) - \overline{\Delta A}_{\min}^{\sigma-} \bar{x}^+(t) \\ &\quad + \overline{\Delta A}_{\max}^{\sigma-} \bar{x}^-(t) \end{aligned} \quad (29)$$

$$\bar{\varphi}_{B,\min}^\sigma(t) = \overline{\Delta B}_{\min}^\sigma u^+(t) - \overline{\Delta B}_{\max}^\sigma u^-(t) \quad (30)$$

$$\begin{aligned} \underline{\varphi}_{A,\min}^\sigma(t) &= \underline{\Delta A}_{\min}^{\sigma+} \underline{x}^+(t) - \underline{\Delta A}_{\max}^{\sigma+} \underline{x}^-(t) - \underline{\Delta A}_{\min}^{\sigma-} \bar{x}^+(t) \\ &\quad + \underline{\Delta A}_{\max}^{\sigma-} \bar{x}^-(t) \end{aligned} \quad (31)$$

$$\underline{\varphi}_{B,\min}^\sigma(t) = \underline{\Delta B}_{\min}^\sigma u^+(t) - \underline{\Delta B}_{\max}^\sigma u^-(t). \quad (32)$$

Let introduce the upper and lower observation errors  $\bar{e}(t) = \bar{x}(t) - x(t)$  and  $\underline{e}(t) = x(t) - \underline{x}(t)$  and note that from (19),  $x(t)$  can be expressed by two different ways: (i) employing upper estimated premise variables or (ii) employing lower estimated premise variables. To obtain  $\bar{e}(t)$  in (33), we use the first expression of  $x(t)$  given in (19) while to obtain  $\underline{e}(t)$  in (34), the second form of (19) is used. Hence,

$$\begin{aligned} \dot{\bar{e}}(t) &= \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) (A_i^\sigma - L_i^\sigma C^\sigma) \bar{e}(t) - d(t) \\ &\quad + \overline{\Delta A}^\sigma(t) x(t) + \overline{\Delta B}^\sigma(t) u(t) + \bar{\psi}(t) \end{aligned} \quad (33)$$

$$\begin{aligned} \dot{\underline{e}}(t) &= \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) (A_i^\sigma - L_i^\sigma C^\sigma) \underline{e}(t) + d(t) \\ &\quad + \underline{\Delta A}^\sigma(t) x(t) + \underline{\Delta B}^\sigma(t) u(t) + \underline{\psi}(t) \end{aligned} \quad (34)$$

where

$$\begin{aligned} \bar{\psi}(t) &= \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) (L_i^\sigma v(t) + |L_i^\sigma| \bar{V} E_p) + \bar{d} \\ &\quad - \bar{\varphi}_{A,\min}^\sigma(t) - \bar{\varphi}_{B,\min}^\sigma(t) \end{aligned} \quad (35)$$

$$\begin{aligned} \underline{\psi}(t) &= \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) (-L_i^\sigma v(t) + |L_i^\sigma| \bar{V} E_p) - \underline{d} \\ &\quad - \underline{\varphi}_{A,\min}^\sigma(t) - \underline{\varphi}_{B,\min}^\sigma(t), \end{aligned} \quad (36)$$

with  $\bar{\varphi}_{A,\min}^\sigma(t)$ ,  $\bar{\varphi}_{B,\min}^\sigma(t)$ ,  $\underline{\varphi}_{A,\min}^\sigma(t)$  and  $\underline{\varphi}_{B,\min}^\sigma(t)$  defined in (29)-(32).

A three sub-steps T-S interval estimation method is proposed as follows:

- Decouple the state  $x(t)$  from the upper and lower errors dynamics (33)-(34).
- Compute appropriate gains  $L_i^\sigma \in \mathbb{R}^{n \times p}$  for each mode  $\sigma \in \{1, 2, \dots, N\}$  in order to establish the global asymptotic stability of the origin of the considered system (19) coupled with upper error dynamic (33) as well as of the considered system (19) coupled with lower error dynamic (34) through a Lyapunov approach. This sub-step is under the heading of "Stability property".
- Ensure the nonnegativity property of the upper and lower errors dynamics given in (33) and (34) (i.e.  $\bar{e}(t) \geq 0$  and  $\underline{e}(t) \geq 0$  to guarantee the relation  $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$ ). This sub-step is under the heading of "Nonnegativity property".

We achieve the first step by defining the augmented upper and lower vectors as  $\bar{e}_a(t) = [\bar{e}^T(t) \quad x^T(t)]^T$  and  $\underline{e}_a(t) =$

$[\underline{e}^T(t) \ x^T(t)]^T$  from which the following dynamics are obtained with  $\sigma \in \{1, 2, \dots, N\}$  and  $i, j \in \{1, \dots, r\}$ :

$$\dot{\bar{e}}_a(t) = \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) \sum_{j=1}^r \mu_j^\sigma(x(t)) (\bar{A}_{ij}^\sigma(t) \bar{e}_a(t) + \bar{B}_{ij}^\sigma(t) u(t) + \bar{F}d(t) + G\bar{\psi}(t)) \quad (37)$$

$$\dot{\underline{e}}_a(t) = \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) \sum_{j=1}^r \mu_j^\sigma(x(t)) (\underline{A}_{ij}^\sigma(t) \underline{e}_a(t) + \underline{B}_{ij}^\sigma(t) u(t) + \underline{F}d(t) + G\underline{\psi}(t)) \quad (38)$$

where

$$\begin{cases} \bar{A}_{ij}^\sigma(t) = \begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \bar{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix}, \bar{B}_{ij}^\sigma = \begin{bmatrix} \bar{\Delta B}^\sigma(t) \\ B_j^\sigma \end{bmatrix} \\ \underline{A}_{ij}^\sigma(t) = \begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \underline{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix}, \underline{B}_{ij}^\sigma = \begin{bmatrix} \underline{\Delta B}^\sigma(t) \\ B_j^\sigma \end{bmatrix} \\ \bar{F} = \begin{bmatrix} -\bar{I} \\ I \end{bmatrix}, \underline{F} = \begin{bmatrix} I \\ I \end{bmatrix}, G = \begin{bmatrix} I \\ 0 \end{bmatrix}. \end{cases}$$

The second and third steps of above-mentioned three-step T-S interval estimation method are respectively attained by the upcoming LMIs (39) and (40) of Theorem 1.

*Theorem 1:* Let the system (19) satisfy Assumptions 1-3 and assume that  $\underline{x}(0)$ ,  $\bar{x}(0)$  are known and the initial state  $x(0)$  verifies  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ . If there exist diagonal positive matrix  $P_1 \in \mathfrak{R}^{n \times n}$ , positive definite matrix  $P_2 \in \mathfrak{R}^{n \times n}$ , matrices  $K^\sigma \in \mathfrak{R}^{n \times p}$  and the strictly positive scalars  $\eta_i^\sigma$ ,  $\rho_1^\sigma$  and  $\lambda^\sigma$  for all  $\sigma \in \{1, \dots, N\}$  such that for all  $i, j \in \{1, \dots, r\}$ ,

$$\begin{bmatrix} \phi_i^\sigma & 0 & P_1 \mathcal{A}^\sigma \\ 0 & \Upsilon_j^\sigma & 0 \\ \mathcal{A}^{\sigma T} P_1 & 0 & -\frac{1}{\rho_1^\sigma} I \end{bmatrix} < 0 \quad (39)$$

$$P_1 A_i^\sigma - K_i^\sigma C^\sigma + \eta_i^\sigma P_1 \geq 0 \quad (40)$$

where

$$\phi_i^\sigma = A_i^{\sigma T} P_1 + P_1 A_i^\sigma - C^{\sigma T} K_i^{\sigma T} - K_i^\sigma C^\sigma + \frac{3}{\lambda^\sigma} P_1$$

$$K_i^\sigma = P_1 L_i^\sigma$$

$$\Upsilon_j^\sigma = A_j^{\sigma T} P_2 + P_2 A_j^\sigma + \frac{3}{\lambda^\sigma} P_2 + \rho_1^\sigma E_A^T E_A$$

with  $\mathcal{A}$ ,  $E_A$  defined in (21), hold, then, (28) is an interval observer for the system (17).

*proof 1:*

1) Stability property

Consider the following common Lyapunov function for the augmented upper dynamic (37):

$$V(\bar{e}_a(t)) = \bar{e}_a^T(t) P \bar{e}_a(t), \quad P = \text{diag}([P_1 \ P_2]) \succ 0 \quad (41)$$

Taking the derivative of the Lyapunov function (63) along all trajectories of (37), then  $\forall \sigma \in \{1, 2, \dots, N\}$  and  $\forall i, j \in \{1, \dots, r\}$ :

$$\begin{aligned} \dot{V}(\bar{e}_a(t)) &= \dot{\bar{e}}_a^T(t) P \bar{e}_a(t) + \bar{e}_a^T(t) P \dot{\bar{e}}_a(t) \\ &= \sum_{i=1}^r \mu_i^\sigma(\bar{x}(t)) \sum_{j=1}^r \mu_j^\sigma(x(t)) (\bar{e}_a^T(t) \bar{A}_{ij}^{\sigma T}(t) P \bar{e}_a(t) \\ &\quad + \bar{e}_a^T(t) P \bar{A}_{ij}^\sigma(t) \bar{e}_a(t) + 2\bar{e}_a^T(t) P \bar{B}_{ij}^\sigma(t) u(t)) \\ &\quad + 2\bar{e}_a^T(t) P \bar{F} d(t) + 2\bar{e}_a^T(t) P G \bar{\psi}(t) \end{aligned} \quad (42)$$

Based on Lemma 5, the following inequalities are deduced where  $\lambda^\sigma > 0$  for all  $\sigma \in \{1, \dots, N\}$  can be selected arbitrarily

$$\begin{aligned} 2\bar{e}_a^T(t) P \bar{B}_{ij}^\sigma(t) u(t) &\leq \frac{1}{\lambda^\sigma} \bar{e}_a^T(t) P \bar{e}_a(t) \\ &\quad + u^T(t) \bar{B}_{ij}^{\sigma T}(t) [\lambda^\sigma P] \bar{B}_{ij}^\sigma(t) u(t) \\ 2\bar{e}_a^T(t) P \bar{F} d(t) &\leq \frac{1}{\lambda^\sigma} \bar{e}_a^T(t) P \bar{e}_a(t) \\ &\quad + d^T(t) \bar{F}^T [\lambda^\sigma P] \bar{F} d(t) \\ 2\bar{e}_a^T(t) P G \bar{\psi}(t) &\leq \frac{1}{\lambda^\sigma} \bar{e}_a^T(t) P \bar{e}_a(t) \\ &\quad + \bar{\psi}^T(t) G^T [\lambda^\sigma P] G \bar{\psi}(t) \end{aligned} \quad (43)$$

The combination of (42) and (43) leads to:

$$\begin{aligned} \dot{V}(\bar{e}_a(t)) &\leq \bar{e}_a^T(t) \sum_{i=1}^r \sum_{j=1}^r \mu_i^\sigma(\bar{x}(t)) \mu_j^\sigma(x(t)) (\bar{A}_{ij}^{\sigma T} P + P \bar{A}_{ij}^\sigma \\ &\quad + \frac{3}{\lambda^\sigma} P) \bar{e}_a(t) + v^\sigma \end{aligned} \quad (44)$$

where for  $\sigma \in \{1, \dots, N\}$ ,  $i, j \in \{1, \dots, r\}$

$$\begin{aligned} v^\sigma &= \\ &\sum_{i=1}^r \sum_{j=1}^r \mu_i^\sigma(\bar{x}(t)) \mu_j^\sigma(x(t)) (u^T(t) \bar{B}_{ij}^{\sigma T}(t) [\lambda^\sigma P] \bar{B}_{ij}^\sigma(t) u(t)) \\ &\quad + d^T(t) \bar{F}^T [\lambda^\sigma P] \bar{F} d(t) + \bar{\psi}^T(t) G^T [\lambda^\sigma P] G \bar{\psi}(t). \end{aligned} \quad (45)$$

According to the convex sum property of the weighting functions  $\mu$  in (15) and from the Lyapunov-based stability analysis for the T-S fuzzy systems [52], to achieve the input-to-state stability (see Lemma 6), we need to prove that

$$\Gamma^\sigma = \bar{A}_{ij}^{\sigma T} P + P \bar{A}_{ij}^\sigma + \frac{3}{\lambda^\sigma} P < 0. \quad (46)$$

Recall that  $P = \text{diag}([P_1 \ P_2])$ , from (46) we have

$$\begin{aligned} \Gamma^\sigma &= \begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T & 0 \\ \bar{\Delta A}^{\sigma T}(t) & A_j^{\sigma T} \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &+ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_i^\sigma - L_i^\sigma C^\sigma & \bar{\Delta A}^\sigma(t) \\ 0 & A_j^\sigma \end{bmatrix} \\ &+ \begin{bmatrix} \frac{3}{\lambda^\sigma} P_1 & 0 \\ 0 & \frac{3}{\lambda^\sigma} P_2 \end{bmatrix}. \end{aligned} \quad (47)$$

Then, it follows

$$\begin{aligned} \Gamma^\sigma &= \begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T P_1 + P_1 (A_i^\sigma - L_i^\sigma C^\sigma) + \frac{3}{\lambda^\sigma} P_1 \\ \bar{\Delta A}^{\sigma T}(t) P_1 \\ P_1 \bar{\Delta A}^\sigma(t) \\ A_j^{\sigma T} P_2 + P_2 A_j^\sigma + \frac{3}{\lambda^\sigma} P_2 \end{bmatrix}. \end{aligned} \quad (48)$$

Let rewrite (48) by separating the time-dependent term  $P_1 \bar{\Delta A}^\sigma(t)$ , we obtain

$$\begin{aligned} \Gamma^\sigma &= \begin{bmatrix} (A_i^\sigma - L_i^\sigma C^\sigma)^T P_1 + P_1 (A_i^\sigma - L_i^\sigma C^\sigma) + \frac{3}{\lambda^\sigma} P_1 \\ 0 \\ 0 \\ A_j^{\sigma T} P_2 + P_2 A_j^\sigma + \frac{3}{\lambda^\sigma} P_2 \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} 0 & P_1 \bar{\Delta A}^\sigma(t) \\ \bar{\Delta A}^{\sigma T}(t) P_1 & 0 \end{bmatrix}}_w. \end{aligned} \quad (49)$$

The matrix  $\mathcal{W}$  can be decomposed such that  $\mathcal{W} = \mathcal{Q} + \mathcal{Q}^T$  where

$$\mathcal{Q} = \begin{bmatrix} 0 & P_1 \overline{\Delta A}^\sigma(t) \\ 0 & 0 \end{bmatrix}. \quad (50)$$

Using the definition of the uncertainty  $\overline{\Delta A}^\sigma(t)$  given in (20), it yields the following partition of  $\mathcal{Q}$

$$\mathcal{Q} = \begin{bmatrix} \overbrace{\begin{bmatrix} P_1 \mathcal{A}^\sigma & 0 \\ 0 & 0 \end{bmatrix}}^X & \overbrace{\begin{bmatrix} 0 & \overline{\Sigma}_A^\sigma(t) E_A \\ 0 & 0 \end{bmatrix}}^Y \end{bmatrix}. \quad (51)$$

Choosing  $\Omega = \text{diag}([\underbrace{\rho_1^\sigma \dots \rho_1^\sigma}_{n \text{ terms}} \dots \underbrace{\rho_2^\sigma \dots \rho_2^\sigma}_{n \text{ terms}}]) \succ 0$  with  $\rho_1^\sigma, \rho_2^\sigma$  are any strictly positive scalars for all  $\sigma \in \{1, \dots, N\}$ . Applying Lemma 4 to (51) yields

$$\mathcal{W} \leq X \Omega^{-1} X^T + Y^T \Omega Y. \quad (52)$$

Bearing in mind that  $\overline{\Sigma}_A^{\sigma T}(t) \overline{\Sigma}_A^\sigma(t) \leq I_{nr}$  (see Remark 1), the following inequality holds

$$\mathcal{W} \leq \text{diag}([\frac{1}{\rho_1^\sigma} P_1 \mathcal{A}^\sigma \mathcal{A}^{\sigma T} P_1 \quad \rho_1^\sigma E_A^T E_A]) \quad (53)$$

Substituting (53) in (49) leads to:

$$\Gamma^\sigma = \text{diag}([\Xi_i^\sigma \quad \Upsilon_j^\sigma]) \quad (54)$$

where

$$\begin{aligned} \Xi_i^\sigma &= (A_i^\sigma - L_i^\sigma C^\sigma)^T P_1 + P_1 (A_i^\sigma - L_i^\sigma C^\sigma) \\ &\quad + \frac{3}{\lambda^\sigma} P_1 + \frac{1}{\rho_1^\sigma} P_1 \mathcal{A}^\sigma \mathcal{A}^{\sigma T} P_1 \\ \Upsilon_j^\sigma &= A_j^{\sigma T} P_2 + P_2 A_j^\sigma + \frac{3}{\lambda^\sigma} P_2 + \rho_1^\sigma E_A^T E_A \end{aligned}$$

From LMI (39), based on the Schur complement [47] with  $K_i^\sigma = P_1 L_i^\sigma$  we can conclude that  $\Gamma^\sigma \prec 0$  for all  $\sigma \in \{1, \dots, N\}$ . On the other hand, because the state  $x(t)$  and measurement noises  $v(t)$  are all bounded in norm (see Assumptions 1-2 which are accepted in many applications),  $\overline{\psi}(t)$  given in (35) is bounded. Moreover the known input  $u(t)$  and additive disturbances  $d(t)$  are also bounded in norm as stated in Assumption 1 and 2, it follows that  $v^\sigma$  in (44) is bounded for all  $\sigma \in \{1, \dots, N\}$ . Thus, from (44) the augmented upper dynamic (37) is ISS due to Lemma 6. Similarly one can prove that the augmented lower dynamic (38) is ISS. Hence, the stability property of the interval observer (28) can be deduced since  $|\overline{x} - \underline{x}| \leq \underbrace{|\overline{x} - x|}_{=\overline{e}} + \underbrace{|x - \underline{x}|}_{=e}$ .

## 2 Nonnegativity property

First, from (29)-(32) and Lemma 2, the following inequalities hold

$$\begin{aligned} \overline{\Delta A}^\sigma(t)x(t) &\geq \overline{\varphi}_{A,\min}^\sigma(t), \quad \underline{\Delta A}^\sigma(t)x(t) \geq \underline{\varphi}_{A,\min}^\sigma(t) \\ \overline{\Delta B}^\sigma(t)u(t) &\geq \overline{\varphi}_{B,\min}^\sigma(t), \quad \underline{\Delta B}^\sigma(t)u(t) \geq \underline{\varphi}_{B,\min}^\sigma(t) \end{aligned} \quad (55)$$

From Assumption 1, we have for all  $\sigma \in \{1, \dots, N\}$ ,  $i \in \{1, \dots, r\}$ ,  $\overline{d} - d \geq 0$ ,  $d - \underline{d} \geq 0$ ,  $L_i^\sigma v(t) + |L_i^\sigma| \overline{V} E_p \geq 0$  and  $-L_i^\sigma v(t) + |L_i^\sigma| \overline{V} E_p \geq 0$ . Thus from (35)-(36), it holds that

$$\begin{aligned} \overline{\psi}(t) - d(t) + \overline{\Delta A}^\sigma(t)x(t) + \overline{\Delta B}^\sigma(t)u(t) &\geq 0 \\ d(t) + \underline{\Delta A}^\sigma(t)x(t) + \underline{\Delta B}^\sigma(t)u(t) + \underline{\psi}(t) &\geq 0 \end{aligned} \quad (56)$$

Subsequently, thanks to (40) and Lemma 3, one can ensure that  $(A_i^\sigma - L_i^\sigma C^\sigma)$  is Metzler for all  $\sigma \in \{1, \dots, N\}$ , for all  $i \in \{1, \dots, r\}$  since  $P_1(A_i^\sigma - L_i^\sigma C^\sigma) + \eta_i^\sigma P_1 \geq 0, \forall \sigma \in \{1, \dots, N\}, \forall i \in \{1, \dots, r\}$  and  $P_1$  is diagonal positive matrix.

Lastly, according to Lemma 1 and bearing in mind the Metzler property of  $(A_i^\sigma - L_i^\sigma C^\sigma)$  together with (56), if  $\overline{x}(0)$  and  $\underline{x}(0)$  are supposed to be known such that

$$\begin{cases} \overline{e}(0) = \overline{x}(0) - x(0) \geq 0 \\ \underline{e}(0) = x(0) - \underline{x}(0) \geq 0 \end{cases},$$

then the dynamics of the estimation errors given in (33)-(34) stay positive and consequently,  $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$  which completes the proof.

## B. Optimal T-S interval design via $H_\infty$ approach with respect to additive disturbances

In this section, we consider that measurement noises are not present (i.e.,  $v(t) = 0$ ). The main objective is to adapt the design of the T-S interval observer (28) in order to estimate an ultimate-bound guaranteeing a tighter interval width which represents the fourth step of this paper. In fact, this section is devoted to the computation of gains  $L_i^\sigma$ ,  $\sigma = \{1, \dots, N\}$ ,  $i \in \{1, \dots, r\}$  to ensure stability properties. Notice that these gains decide also the tightness of the interval width. Hence, the goal is not only to ensure the stability of the ultimate-bound as in Section III-A but also to improve the accuracy of the T-S interval observer (28) and the idea is based on an  $H_\infty$  approach. The effect of the known bound of the additive disturbance  $d$  on the estimation error is reduced by the observer gain matrices. In other words, we are interested in computing observer gains  $L_i^\sigma$  which minimize a cost function to be introduced later.

Given the structure of the T-S interval observer (28), by taking account  $\overline{V} = v(t) = 0$ , (33)-(34) can be rewritten as follows

$$\begin{aligned} \dot{\overline{e}}(t) &= \sum_{i=1}^r \mu_i^\sigma(\overline{x}(t)) ((A_i^\sigma - L_i^\sigma C^\sigma) \overline{e}(t) + \overline{d} - d(t) \\ &\quad + \overline{\Delta A}^\sigma(t)x(t) + \overline{\Delta B}^\sigma(t)u(t) + \overline{\psi}^*(t)) \end{aligned} \quad (57)$$

$$\begin{aligned} \dot{\underline{e}}(t) &= \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) ((A_i^\sigma - L_i^\sigma C^\sigma) \underline{e}(t) + d(t) - \underline{d} \\ &\quad + \underline{\Delta A}^\sigma(t)x(t) + \underline{\Delta B}^\sigma(t)u(t) + \underline{\psi}^*(t)) \end{aligned} \quad (58)$$

where  $\overline{\psi}^*(t) = -\overline{\varphi}_{A,\min}^\sigma(t) - \overline{\varphi}_{B,\min}^\sigma(t)$  and  $\underline{\psi}^*(t) = -\underline{\varphi}_{A,\min}^\sigma(t) - \underline{\varphi}_{B,\min}^\sigma(t)$  with  $\overline{\varphi}_{A,\min}^\sigma(t), \overline{\varphi}_{B,\min}^\sigma(t), \underline{\varphi}_{A,\min}^\sigma(t)$  and  $\underline{\varphi}_{B,\min}^\sigma(t)$  defined in (29)-(32).

Analogously to (37)-(38) by defining  $\overline{e}_a(t) = [\overline{e}^T(t) \quad x^T(t)]^T$  and  $\underline{e}_a(t) = [\underline{e}^T(t) \quad x^T(t)]^T$ , the following dynamics are obtained with  $\sigma \in \{1, 2, \dots, N\}$  and  $i, j \in \{1, \dots, r\}$ :

$$\begin{aligned} \dot{\overline{e}}_a(t) &= \sum_{i=1}^r \mu_i^\sigma(\overline{x}(t)) \sum_{j=1}^r \mu_j^\sigma(x(t)) (\overline{A}_{ij}^\sigma(t) \overline{e}_a(t) \\ &\quad + \overline{B}_{ij}^\sigma(t)u(t)) + \overline{E} \overline{d} + \overline{F} d(t) + \overline{G} \overline{\psi}^*(t) \end{aligned} \quad (59)$$



$$\begin{aligned} \dot{\bar{e}}_a(t) = & \sum_{i=1}^r \mu_i^\sigma(\underline{x}(t)) \sum_{j=1}^r \mu_j^\sigma(x(t)) (\underline{A}_{ij}^\sigma(t) \bar{e}_a(t) \\ & + \underline{B}_{ij}^\sigma(t) u(t)) + \underline{F}d(t) + \underline{E}d + G\underline{\psi}^*(t) \end{aligned} \quad (60)$$

where  $\bar{E} = [I \ 0]^T$ ,  $\underline{E} = [-I \ 0]^T$ ,  $\bar{F} = [-I \ I]^T$ ,  $\underline{F} = [I \ I]^T$ .

*Theorem 2:* Let the system (19) satisfy Assumptions 1-3 and assume that  $\underline{x}(0)$ ,  $\bar{x}(0)$  are known and the initial state  $x(0)$  verifies  $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$ . If there exist diagonal positive matrix  $P_1 \in \mathbb{R}^{n \times n}$ , positive definite matrix  $P_2 \in \mathbb{R}^{n \times n}$ , matrices  $K^\sigma \in \mathbb{R}^{n \times p}$  and the strictly positive scalars  $\eta_i^\sigma$ ,  $\rho_1^\sigma$ ,  $\bar{\gamma}$  and  $\lambda^\sigma$  for all  $\sigma \in \{1, \dots, N\}$  such that for all  $i, j \in \{1, \dots, r\}$ , the following constrained minimization problem

$$\begin{aligned} & \underset{P_1, P_2, K_i^\sigma, \rho_1^\sigma}{\text{minimize}} \quad \bar{\gamma} \\ & \text{subject to} \quad \begin{bmatrix} \phi_i^\sigma & 0 & P_1 & P_1 A^\sigma \\ 0 & \Upsilon_j^\sigma & 0 & 0 \\ P_1 & 0 & -\bar{\gamma}I & 0 \\ A^{\sigma T} P & 0 & 0 & -\frac{1}{\rho_1^\sigma} I \end{bmatrix} < 0 \\ & P_1 A_i^\sigma - K_i^\sigma C^\sigma + \eta_i^\sigma P_1 \geq 0. \end{aligned} \quad (61)$$

where

$$\begin{aligned} \phi_i^\sigma &= A_i^{\sigma T} P_1 + P_1 A_i^\sigma - C^{\sigma T} K_i^{\sigma T} - K_i^\sigma C^\sigma + \frac{3}{\lambda^\sigma} P_1 + I_n \\ K_i^\sigma &= P_1 L_i^\sigma \\ \Upsilon_j^\sigma &= A_j^{\sigma T} P_2 + P_2 A_j^\sigma + \frac{3}{\lambda^\sigma} P_2 + \rho_1^\sigma E_A^T E_A, \end{aligned}$$

with  $A$ ,  $E_A$  defined in (21), is solved, then when  $v(t) = \bar{V} = 0$ , (28) is an optimal interval observer for the system (17) that guarantees the attenuation of additive disturbances effect with the cost function computed by

$$\gamma = \sqrt{\bar{\gamma}}. \quad (62)$$

*Remark 3:* Notice that the terms  $\eta_i^\sigma$  are fixed before solving the LMIs (40) and (61). Thus, those LMIs are not nonlinear optimization problems.

*proof 2:*

### 1) Stability property

Employing the following common Lyapunov function for the augmented upper dynamic (59):

$$V(\bar{e}_a(t)) = \bar{e}_a^T(t) P \bar{e}_a(t), \quad P = \text{diag}([P_1 \ P_2]) \succ 0 \quad (63)$$

to prove that (59) is ISS. Indeed, in the same way of the proof of Theorem 1 given in (42)-(44), it follows that

$$\begin{aligned} \dot{V}(\bar{e}_a(t)) \leq & \bar{e}_a^T(t) \left( \sum_{i=1}^r \sum_{j=1}^r \mu_i^\sigma(\bar{x}(t)) \mu_j^\sigma(x(t)) \Gamma^\sigma \right) \bar{e}_a(t) \\ & + d^T \bar{E}^T P \bar{e}_a(t) + \bar{e}_a^T P \bar{E} d + \vartheta^\sigma \end{aligned} \quad (64)$$

where

$$\begin{aligned} \vartheta^\sigma = & \sum_{i=1}^r \sum_{j=1}^r \mu_i^\sigma(\bar{x}(t)) \mu_j^\sigma(x(t)) \left( u^T(t) \bar{B}_{ij}^{\sigma T}(t) [\lambda^\sigma P] \bar{B}_{ij}^\sigma(t) u(t) \right) \\ & + d^T(t) \bar{F}^T [\lambda^\sigma P] \bar{F} d(t) + \bar{\psi}^{*T}(t) G^T [\lambda^\sigma P] G \bar{\psi}^*(t) \end{aligned} \quad (65)$$

with  $\bar{\psi}^*$  defined in (57).

Besides, the upper estimation error given in (57) can be seen as

$$\bar{e}(t) = H \bar{e}_a(t) \quad (66)$$

where  $H = [I \ 0]$ .

Based on  $H_\infty$  approach, the optimal gains are computed in order to minimize the following cost function  $\forall \sigma \in \{1, 2, \dots, N\}$  and  $\forall i, j \in \{1, \dots, r\}$ :

$$\begin{aligned} & \underset{L_i^\sigma \in \mathbb{R}^{n \times p}}{\text{minimize}} \quad \gamma^2 \\ & \text{subject to} \quad \frac{\|\bar{e}(t)\|_2}{\|\bar{d}\|_2} \leq \gamma^2 \end{aligned} \quad (67)$$

The effect of the known bound of the additive disturbances  $d(t)$  on the upper observation error  $\bar{e}$  is reduced by the observer gain matrix  $L_i^\sigma$  which are computed via the minimization of the positive real number  $\bar{\gamma} = \gamma^2$ .

The optimization problem (67) can be reformulated under an LMI form using the bounded-real lemma [47] for (57) which leads to:

$$\dot{V}(\bar{e}_a(t)) + \bar{e}^T(t) \bar{e}(t) - \bar{\gamma} d^T d \leq 0 \quad (68)$$

The same reasoning of the proof of Theorem 1 applies, we deduce that  $\vartheta^\sigma$  given in (65) is bounded. Thus, by substituting (64) and (66) in (68), (59) is ISS if the following inequality holds

$$\begin{aligned} \bar{e}_a^T(t) \Gamma^\sigma \bar{e}_a(t) + d^T \bar{E}^T P \bar{e}_a(t) + \bar{e}_a^T P \bar{E} d \\ + \bar{e}_a^T(t) H^T H \bar{e}_a - \bar{\gamma} d^T d \leq 0 \end{aligned} \quad (69)$$

or equivalently

$$\begin{bmatrix} \bar{e}_a \\ \bar{d} \end{bmatrix}^T \begin{bmatrix} \Gamma^\sigma + H^T H & P \bar{E} \\ \bar{E}^T P & -\bar{\gamma} I \end{bmatrix} \begin{bmatrix} \bar{e}_a \\ \bar{d} \end{bmatrix} \leq 0 \quad (70)$$

Replacing the term  $\Gamma^\sigma$  by its expression given in (54) and from the definition of  $H$  in (66) we have  $H^T H = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$ , the inequality (70) holds if the subsequent one is satisfied

$$\begin{bmatrix} \Xi_i^\sigma + I_n & 0 & P \bar{E} \\ 0 & \Upsilon_j^\sigma & 0 \\ \bar{E}^T P & 0 & -\bar{\gamma} I \end{bmatrix} < 0 \quad (71)$$

From the LMI (61), based on the Schur complement [47] with  $K_i^\sigma = P_1 L_i^\sigma$ , this allows us to conclude. Similarly one can prove that the augmented lower dynamic (60) is ISS. Hence, the stability property of the interval observer (28) can be deduced since  $|\bar{x} - x| \leq \underbrace{|\bar{x} - \underline{x}|}_{=\bar{e}} + \underbrace{|x - \underline{x}|}_{=e}$ .

### 2 Nonnegativity property

This proof is identical to the one given in Theorem 1.

*Remark 4:*

- LMIs given in Theorem 1 and Theorem 2 ensure at the same time the ISS and nonnegativity properties of T-S interval observer (28). In addition, minimizing cost function in Theorem 2 yields optimal gains that improve the accuracy of the present T-S interval observer.

- The presence of measurement noises in (61) leads to a nonlinear optimization problem which cannot be easily solved. This is the design constraints of  $H_\infty$ -based T-S interval observer (28). We conjecture that one can overcome these design constraints by proposing another guaranteed state estimation approach, e.g., zonotopic technique paired with  $H_\infty$  design which is an interesting perspective for the future work. Furthermore, it is worth highlighting that in the presence of  $v(t)$ , Theorem 1 works and T-S interval observer (28) can be always designed.

### C. Application to robust fault detection

In this section, the previous results are used to generate residuals for fault detection. Under the presence of a sensor fault and the absence of measurement noises ( $v(t) = 0$ ), system (17) can be represented by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i^\sigma(x(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + f(t) \\ \forall \sigma \in \{1, 2, \dots, N\}, \forall i \in \{1, \dots, r\}, \end{cases} \quad (72)$$

where  $f(t) \in \mathbb{R}^p$  denotes  $p^{\text{th}}$  sensor fault. The principle is to compare the measurements  $y(t)$  with their estimates  $\hat{y}(t)$  provided by a faultless model. The comparison leads to the generation of a residual  $r(t) \in \mathbb{R}^p$  given by:

$$r(t) = \hat{y}(t) - y(t). \quad (73)$$

In a fault-free operation, the residual are around zero. Nevertheless, when considering a system affected by perturbations and uncertainties given in (19), the residuals deviate from zero even in the fault-free scenario. To cope with this problem, a passive approach is used based on the interval observer (28) designed in the previous section.

Based on (2) of Lemma 2, the lower and upper outputs of the system (17) are given by:

$$\begin{cases} \underline{y}(t) = C^{\sigma+} \underline{x}(t) - C^{\sigma-} \underline{x}(t) \\ \bar{y}(t) = C^{\sigma+} \bar{x}(t) - C^{\sigma-} \bar{x}(t) \end{cases} \quad (74)$$

Let  $[\underline{y}(t), \bar{y}(t)]$  be the domain of the output  $y(t)$ , the fault detection test can be formulated as  $y(t) \notin [\underline{y}(t), \bar{y}(t)]$  which is equivalent to:

$$0 \notin [\underline{r}(t), \bar{r}(t)]. \quad (75)$$

where

$$\begin{cases} \underline{r}(t) = \underline{y}(t) - y(t) = -C^{\sigma+}(x(t) - \underline{x}(t)) \\ \quad \quad \quad \quad \quad \quad \quad \quad -C^{\sigma-}(\bar{x}(t) - x(t)) \\ \bar{r}(t) = \bar{y}(t) - y(t) = C^{\sigma+}(\bar{x}(t) - x(t)) \\ \quad \quad \quad \quad \quad \quad \quad \quad -C^{\sigma-}(x(t) - \underline{x}(t)) \end{cases} \quad (76)$$

Thus, the residual is described by an adaptive threshold.

*Remark 5:* The considered system in this paper is affected by unknown but bounded disturbances. If these bounds are large, so does the width of the interval observer and it may lead to misdetection of small faults. The proposed T-S interval observer design method given in (28) allows to compute optimal gains which attenuate the effect of the system's disturbances

and ensure a tighter interval width which make it possible to detect low magnitude faults.

## IV. NUMERICAL SIMULATIONS

### A. Example 1

In order to illustrate the effectiveness of the proposed approaches, Theorem 1 is firstly illustrated and secondly, a comparative study is made between the results obtained in Theorem 1 and those optimized in Theorem 2 in the absence of measurement noises.

Let consider a switched system characterized by two nonlinear modes (i.e.  $\sigma \in \{1, 2\}$ ) where each mode is represented by T-S fuzzy models with two local models (i.e.  $r = 2$ ):

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^2 \mu_i^\sigma(\xi(t))(A_i^\sigma x(t) + B_i^\sigma u(t)) + d(t) \\ y(t) = C^\sigma x(t) + v(t) \end{cases} \quad (77)$$

where

For Mode  $\sigma = 1$ :

$$\begin{aligned} A_1^1 &= \begin{bmatrix} -1.51 & -0.262 \\ 0 & -0.1 \end{bmatrix}, A_2^1 = \begin{bmatrix} -0.86 & 1.47 \\ 0 & -0.15 \end{bmatrix} \\ B_1^1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_2^1 = B_1^1, C^1 = [0 \quad 1.5] \end{aligned} \quad (78)$$

For Mode  $\sigma = 2$ :

$$\begin{aligned} A_1^2 &= \begin{bmatrix} -5.55 & 4.1 \\ 0 & -0.1 \end{bmatrix}, A_2^2 = \begin{bmatrix} -2.65 & 2.34 \\ 0 & -0.15 \end{bmatrix} \\ B_1^2 &= \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, B_2^2 = B_1^2, C^2 = [0 \quad 1.5] \end{aligned} \quad (79)$$

The weighting functions are hyperbolic tangent functions and depend on the state of the switched system such as

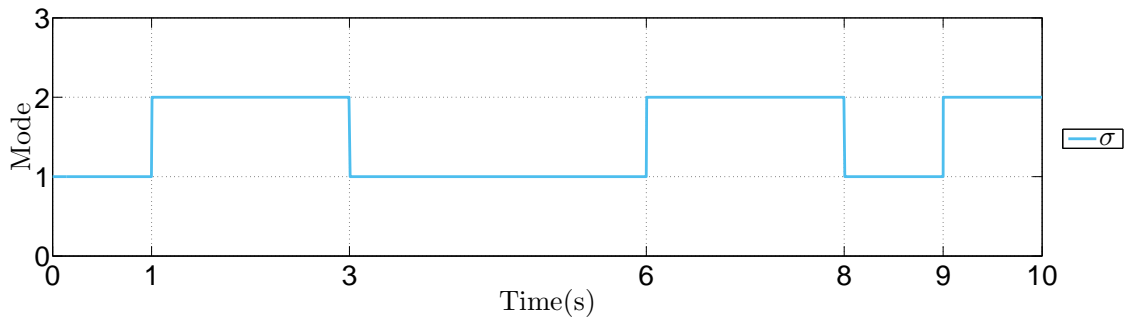
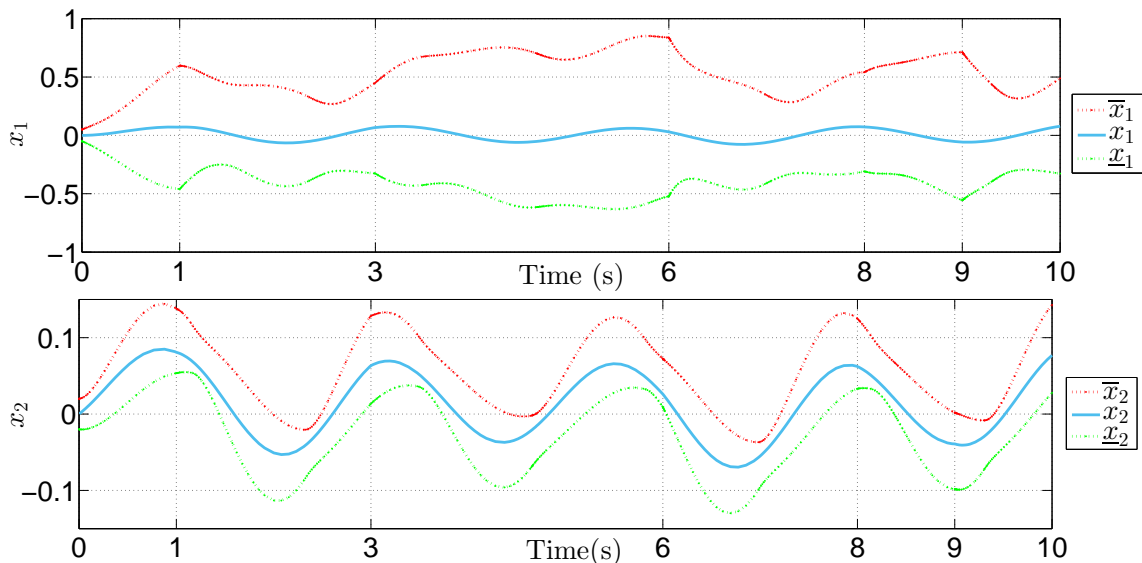
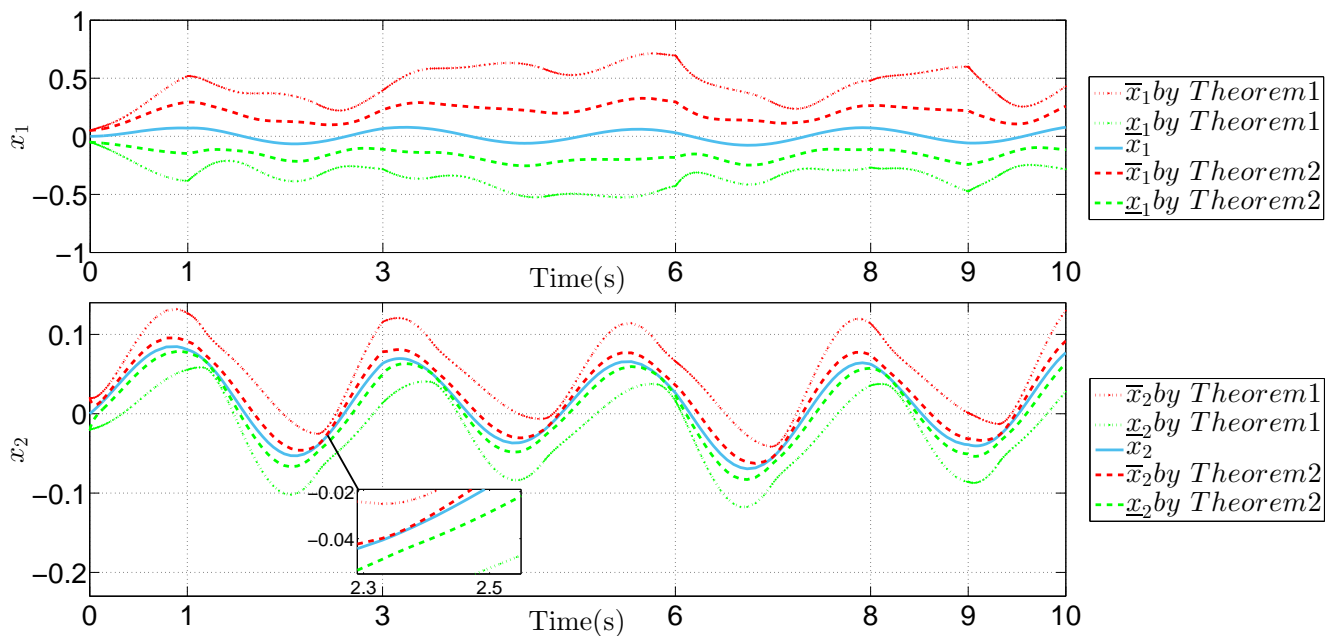
$$\begin{cases} \xi(t) = x_1(t) \\ \mu_1^\sigma(x(t)) = \frac{1}{2}(1 - \tanh(x_1(t))), \forall \sigma \in \{1, 2\} \\ \mu_2^\sigma(x(t)) = 1 - \mu_1^\sigma(x_1(t)), \forall \sigma \in \{1, 2\} \end{cases} \quad (80)$$

For the simulation, the disturbances and the measurement noises are chosen such as:

$$\begin{aligned} d(t) &= 0.1 [\sin(2.7t) \quad \cos(2.7t)]^T \\ \bar{d} = -\underline{d} &= [0.1 \quad 0.1]^T \\ v(t) &= 0.01 \sin(2.7t), \bar{V} = 0.01 \end{aligned} \quad (81)$$

Thus, Assumption 1 is satisfied. The switching signal between the two modes of the considered system is plotted in Figure 2.

The initial conditions are  $x(0) = [0 \quad 0]^T$  and  $\bar{x}(0) = -\underline{x}(0) = [0.05 \quad 0.02]^T$ . In the simulation, the control input is given by  $u(t) = 0.1 \sin(2.7t)$ . The solutions of Theorem 1 and 2 are obtained for  $\lambda^1 = 25.96$ ,  $\lambda^2 = 85.76$ ,  $\eta_1^1 = \eta_2^1 = 10$  and  $\eta_1^2 = \eta_2^2 = 25$ .

Fig. 2. Switching signal  $\sigma$ Fig. 3.  $x_1$ ,  $x_2$  and their interval estimations with measurement noises  $v(t)$  by Theorem 1Fig. 4.  $x_1$ ,  $x_2$  and their interval estimations without measurement noises ( $v(t) = 0$ ) by Theorem 1 and by Theorem 2

1) *Solution of Theorem 1:* Using the package Yalmip toolbox [53], the solution of LMIs (39) of Theorem 1 is given by:

$$\begin{aligned} L_1^1 &= \begin{bmatrix} -3.0735 \\ 2.5767 \end{bmatrix}, L_2^1 = \begin{bmatrix} -0.8078 \\ 2.5981 \end{bmatrix}, \rho_1^1 = 0.2345, \\ L_1^2 &= \begin{bmatrix} -2.4280 \\ 2.4189 \end{bmatrix}, L_2^2 = \begin{bmatrix} -0.8353 \\ 2.5057 \end{bmatrix}, \rho_1^2 = 0.6645, \\ P_1 &= \begin{bmatrix} 0.0425 & 0.0000 \\ 0.0000 & 0.1878 \end{bmatrix}, P_2 = \begin{bmatrix} 0.4463 & 0.0000 \\ 0.0000 & 15.1907 \end{bmatrix}. \end{aligned}$$

We verify that the matrices  $A_i^\sigma - L_i^\sigma C^\sigma$  are Metzler  $\forall \sigma \in \{1, 2\}$ ,  $\forall i \in \{1, 2\}$ . The T-S interval observer (19) is applied. The simulation results given in Figure 3 show that the state stays in the estimated interval all the time, even when the measurement noises and the additive disturbances in (81) are present. In addition, the upper and lower bounds remain stable despite the switching instants. However here the optimization issue of interval width between upper and lower bound is not considered.

2) *Comparisons study between Theorems 1 and 2:* In this section, we consider that the system is affected only by additive disturbances  $d(t)$  (i.e.  $v(t) = 0$ ). Using the package CVX [54], the solution of LMIs (61) of Theorem 2 is given by

$$\begin{aligned} L_1^1 &= \begin{bmatrix} -0.1747 \\ 10.1468 \end{bmatrix}, L_2^1 = \begin{bmatrix} -0.9800 \\ 10.1135 \end{bmatrix}, \rho_1^1 = 1.3062 \cdot 10^6 \\ L_1^2 &= \begin{bmatrix} 2.7333 \\ 10.1334 \end{bmatrix}, L_2^2 = \begin{bmatrix} 1.5600 \\ 10.1000 \end{bmatrix}, \rho_1^2 = 3.5739 \cdot 10^6 \\ P_1 &= \begin{bmatrix} 1.2465 & 0.0000 \\ 0.0000 & 32.8409 \end{bmatrix}, \\ P_2 &= 10^7 \begin{bmatrix} 0.2379 & 0.0000 \\ 0.0000 & 8.3253 \end{bmatrix}. \end{aligned}$$

The resulting attenuation level is  $\bar{\gamma} = 1.2465$ . We verify that the matrices  $A_i^\sigma - L_i^\sigma C^\sigma$  are Metzler  $\forall \sigma \in \{1, 2\}$ ,  $\forall i \in \{1, 2\}$ .

For the purpose of comparison, the simulation of the T-S interval observer (28) with non optimal and optimal gains are depicted in Figure 4 where solid lines present the state and dashed lines present the estimated bounds. Under the same simulation conditions, the results show that the gains computed by Theorem 2 give more accurate interval estimation than the ones computed by Theorem 1 in the absence of measurement noise. Optimal gains offer a tighter interval width. This is explained by the reduction of the effect of the known bound of the additive disturbances on the estimation error. The accuracy of interval observer is then improved.

## B. Example 2

To show the effectiveness of the proposed fault detection method, a switched system described by (72) is considered as follows

$$\begin{aligned} A_1^1 &= \begin{bmatrix} -0.9 & 0 & -0.45 \\ 0 & -2.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\ A_2^1 &= \begin{bmatrix} -3.86 & 0 & 1.22 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} A_1^2 &= \begin{bmatrix} -5.5 & 0 & 1.5 \\ 0 & -1.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\ A_2^2 &= \begin{bmatrix} -2.6 & 0 & 0.3 \\ 0 & -0.15 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \\ B_1^1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\ B_2^1 &= B_2^2 = B_2^3 = B_1^1, \\ C^1 &= [0 \ 0 \ 1.2], \ C^2 = [0 \ 0 \ 1.7]. \end{aligned}$$

The weighting functions are hyperbolic tangent functions and depend on the unmeasured state  $x_1$ :

$$\begin{cases} \xi(t) = x_1(t) \\ \mu_1^\sigma(x(t)) = \frac{1}{2}(1 - \tanh(x_1(t))), \ \forall \sigma \in \{1, 2\} \\ \mu_2^\sigma(x(t)) = 1 - \mu_1^\sigma(x_1(t)), \ \forall \sigma \in \{1, 2\} \end{cases} \quad (82)$$

For the simulation, the disturbances are chosen such as:  $d(t) = 0.1[\cos(3.5t) \ \cos(3.5t) \ \cos(3.5t)]^T$  and  $\bar{d} = -\underline{d} = [0.1 \ 0.1 \ 0.1]^T$ . Thus, Assumption 1 is satisfied. The switching signal between the two modes of the considered system is plotted similarly to Example IV-A in Figure 2. The fault signal is set up as:

$$f(t) = \begin{cases} 0.03, & 2s \leq t \leq 4s \\ 0.02, & 8s \leq t \leq 9s \\ 0 & \text{otherwise} \end{cases} \quad (83)$$

The initial conditions are  $x(0) = [0 \ 0 \ 0]^T$  and  $\bar{x}(0) = -\underline{x}(0) = [0.1 \ 0.1 \ 0.1]^T$ . By fixing  $\lambda^1 = 85.96$ ,  $\lambda^2 = 85.76$ ,  $\eta_1^1 = \eta_2^1 = 10$  and  $\eta_1^2 = \eta_2^2 = 26$ , the solution of LMIs (61) of Theorem 2 are obtained using the package CVX [54]. The values of the optimal gains are given by:

$$\begin{aligned} L_1^1 &= \begin{bmatrix} -0.3750 \\ 0.0000 \\ 16.4867 \end{bmatrix}, L_2^1 = \begin{bmatrix} 1.0167 \\ 0.0000 \\ 16.4867 \end{bmatrix}, \\ L_1^2 &= \begin{bmatrix} 0.0824 \\ 0.0000 \\ 11.6377 \end{bmatrix}, L_2^2 = \begin{bmatrix} 0.1765 \\ 0.0000 \\ 11.6377 \end{bmatrix} \end{aligned}$$

The attenuation level is  $\bar{\gamma} = 7.5443$ . The matrices  $A_i^\sigma - L_i^\sigma C^\sigma$  are Metzler for all  $\sigma \in \{1, 2\}$  and for all  $i \in \{1, 2\}$ . In Figure 5, it is clear that the relations  $\underline{y}(t) \leq y(t) \leq \bar{y}(t)$  and  $0 \in [\underline{r}(t), \bar{r}(t)]$  hold in fault-free case while these relations are broken when the fault occurs. It should be noted that despite the low values of the considered fault (83), the detection is successful. At the instant  $t = 4.01s$ , the fault is still detected, which is explained by the fault extension since the switching instant  $t = 3s$  happens during the faulty period  $2s \leq t \leq 4s$ .

## V. CONCLUSION

In this paper, a new approach is proposed to deal with the problem of interval observer design for nonlinear switched systems described by T-S fuzzy models. The problem is challenging because the premise variables are unmeasurable and the membership functions  $\mu_i$ ,  $i \in \{1, \dots, r\}$  are not assumed to be globally Lipschitz. Thanks to the fact that the

premise variables can be replaced by their interval estimates, this structure allows us to see the considered system as an uncertain system subject to unknown but bounded disturbances/measurement noises. The first goal of the paper is to design T-S interval observer by solving a set of LMIs that ensure at the same time the stability and nonnegativity properties. In the absence of measurement noises, the second goal of the paper is to tighten the width of the present interval observer by pairing it with  $H_\infty$  approach. A sensor fault detection scheme based on interval observers is also given and can be considered as an interesting application. A comparative study between the first and second contribution as well as fault detection issues are successfully done by simulations to illustrate the efficiency of the proposed approaches. Future works will be focused on improving these results when the switching signal is unknown and has to be estimated. On the other hand, the design of closed-loop interval observers which not only gives interval estimates, but also helps to stabilize possibly unstable plants by feedback and adaptive controllers in the spirit of what is done and reviewed in [55]–[57], is also an interesting perspective.

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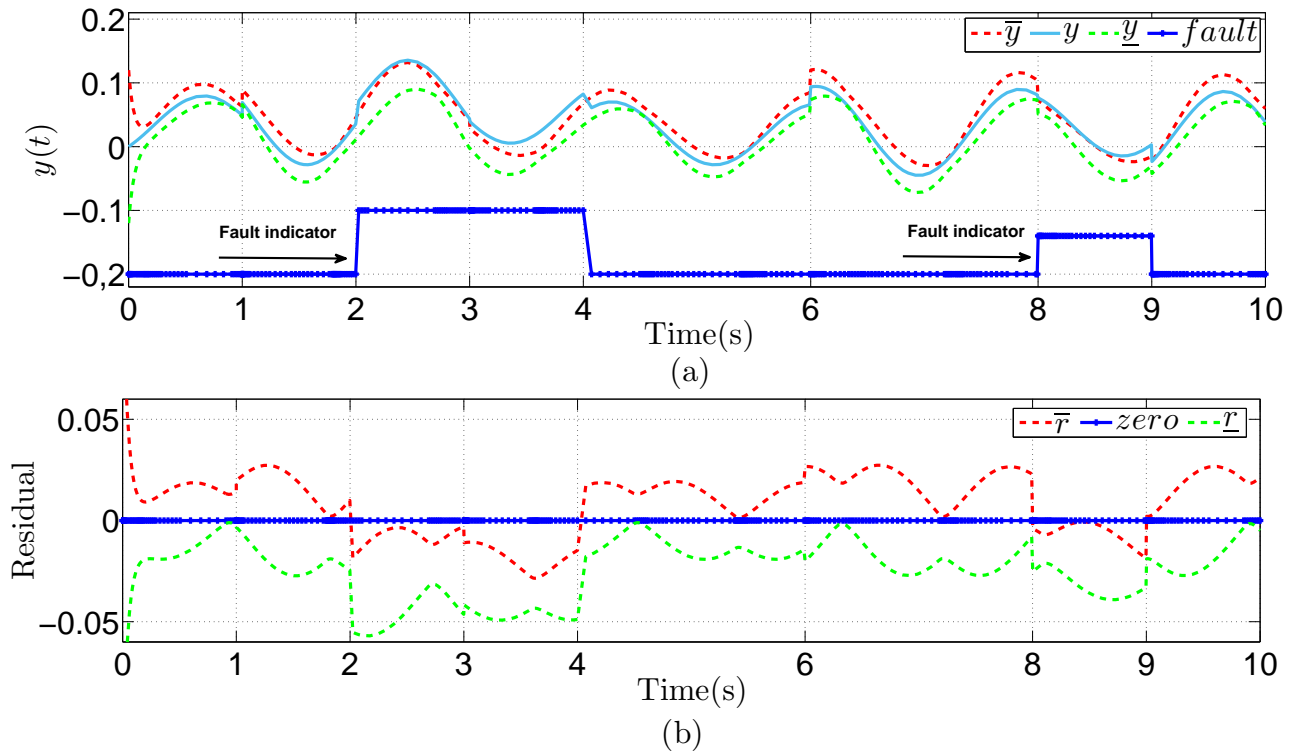


Fig. 5. Interval estimation of (a) the output and (b) the residual

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