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▶ **To cite this version:**

Thierry Horsin, Mohamed Ali Jendoubi. On the convergence to equilibria of a sequence defined by an implicite scheme. 2020. hal-02941522

**HAL Id: hal-02941522**

**https://cnam.hal.science/hal-02941522v1**

Preprint submitted on 17 Sep 2020

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# ON THE CONVERGENCE TO EQUILIBRIA OF A SEQUENCE DEFINED BY AN IMPLICIT SCHEME

THIERRY HORSIN AND MOHAMED ALI JENDOUBI

Abstract. We present numerical implicit scheme based on a geometric approach to the study of the convergence of solutions of gradient-like systems given in [2]. Depending on the globality of the induced metric, we can prove the convergence of these algorithms.

*Dedicated to the memory of Ezzeddine ZAHROUNI*

## 1. Notation

For a riemannian manifold $(M, g)$ of dimension $N$ we denote $\langle \cdot, \cdot \rangle_g$ the scalar product defined on each tangent space. The induced norm is denoted $\| \cdot \|_g$ (or $\| \cdot \|$ when there is no risk of confusion ) For a local system of coordinates on $M$, $g_{ij}$ will denote the coefficient of the matrix defining the scalar product above.

Let us recall that a $C^1$ curve $x : [0, 1] \to M$ is called a geodesic between $x(0)$ and $x(1)$ iff it is a critical point of the functional

$$L(\gamma) = \int_0^1 ||\gamma'(t)||_g dt$$

restricted to the $C^1$-curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x(0)$ and $\gamma(1) = x(1)$.

For a differentiable function $f : M \to \mathbb{R}$ and $p \in M$ we denote $\nabla_g f(p)$ the unique element of the tangent space $T_pM$ to $M$ at $p$ such that

$$\forall u \in T_pM,\ \langle \nabla_g f(p), u \rangle_g = df(p).u$$

## 2. A implicit numerical scheme and main result of the paper

Let us consider $(M, g)$ a complete connected non compact riemaniann manifold and $\mathcal{E}$ a smooth real function. Associated to $\mathcal{E}$, it is quite natural to consider the following gradient system

$$\dot{X}(t) + \nabla_g \mathcal{E}(X(t)) = 0. \tag{1}$$

In the paper [11] the authors Merlet & Pierre consider the situation when $(M, g)$ is the standard $\mathbb{R}^N$ with its natural euclidian structure and prove the convergence of a sequence defined by an implicit scheme associated to (1). It is quite natural to extend the scheme there introduced to the case of more general manifolds. Such insights were initially considered in [12] provided $(M, g)$ is a submanifold of $\mathbb{R}^N$. However the specific case of the backward Euler scheme was not considered in this paper under the intrinsic point of view, *i.e.* the backward scheme is constructed *ex post* in [12], considering the embedded situation. Here we try to focus on the

The first author wishes to thanks the organizers of ICAAM 2019 in Hammamet, Tunisia, where this work was initiated.

The second author wishes to thanks CNAM, France where this work was partially completed.

intrinsic geometry given by $g$ even if we will use the existence of isometric embeddings in some euclidean space. Of course comparing the backward algorithm given in [12] and the scheme constructed in the present paper is certainly of interest but we have not considered it yet for the moment being.

We will assume in this section that $M$ is complete, *i.e.* for any pair of distinct points of $M$ there exists a minimizing geodesic between them. Without loss of much generality, according to Nash's theorem (see [6] and [13]) we can always assume that $M$ is isometrically embedded in $\mathbb{R}^P$ for a large enough $P$. The induced distance on $M$ will be denoted by $d$.

For some $\delta t > 0$, we consider the following sequence : Assume that $X_0$, .., $X_n$ are constructed, we consider

$$X_{n+1} \in \arg\min \frac{d(X, X_n)^2}{2\delta t} + \mathcal{E}(X). \tag{2}$$

The existence and uniqueness of $X_{n+1}$ depends on different hypothesis. A natural assumption is that $\mathcal{E}$ is coercive and semi-convex. From now on, we assume the existence and uniqueness of the sequence $(X_n)$ .

**Definition 2.1.** Provided that for each $n$, $X_n$ is uniquely defined, the sequence $(X_n)$ is the implicit Euler scheme associated to (1), for the given time step $\delta_t$.

The convergence of the solutions of (1) has been extensively studied either in finite or infinite dimensions. In the situation when $\mathcal{E}$ is analytic the convergence was firstly studied by S. Lojasiewicz in [9, 10] (see also [7, 5])

A major sufficient assumption for proving the convergence is the fact that $\mathcal{E}$ satisfies the so-called Lojasiewicz's inequality at critical points:

$$\begin{aligned} &\forall p \in M,\ \nabla_g \mathcal{E}(p) = 0 \Rightarrow \\ &\Big(\exists \theta \in (0, \frac{1}{2}],\ \exists c_p > 0,\ \exists \sigma_p > 0,\ \forall q \in M,\ d(p,q) < \sigma_p \\ &\Rightarrow \|\nabla_g \mathcal{E}(q)\| \geq c_p |\mathcal{E}(p) - \mathcal{E}(q)|^{1-\theta}\Big) \end{aligned} \tag{3}$$

In the following section we will prove the following theorem, main result of this paper.

**Theorem 2.2.** *Assume that $\mathcal{E}$ is coercive and semi-convex and satisfies the Lojasiewicz's inequality then the sequence $(X_n)$ converges to a critical point of $\mathcal{E}$.*

As we said, the convergence of the sequence defined by discretized schemes associated to dynamical systems has been recently studied. The pioneering work in that direction is given in [1]. The case of implicit scheme was considered quite simultaneously in [11] and [3].

In order to deal with this result, it is required to get some informations from the Euler-Lagrange equation satisfied by $X_{n+1}$.

The computation is similar to the one made in order to derive the geodesic equations. We closely follow it (see for example [8]). In order to do so, let

$$\phi_n : [0,1]^2 \to M$$

be a $C^2$ map such that

$$\begin{aligned}&\phi_n(.,0):[0,1]\to M \text{ is a constant speed geodesic from } X_n \text{ to } X_{n+1}\\&\phi_n(0,s)=X_n,\ \forall s\in[0,1].\end{aligned}$$

For any $s\in[0,1]$ we have

$$d(X_n,\phi(1,s))=\int_0^1\left(\sum_{i,j}g_{i,j}(\phi(t,s))\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\right)^{1/2}dt$$

The derivative of $s\mapsto d(X_n,\phi(1,s))$ with respect to $s$ is thus

$$\int_0^1\frac{dt}{2\left(\sum_{i,j}g_{i,j}(\phi(t,s))\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\right)^{1/2}}$$

$$\left(\sum_{i,j}\sum_k\frac{\partial g_{i,j}}{\partial x_k}(\phi(t,s))\frac{\partial\phi_k}{\partial s}\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}+\qquad 2\sum_{i,j}g_{i,j}(\phi(t,s))\frac{\partial^2\phi_i}{\partial t\partial s}\frac{\partial\phi_j}{\partial t}\right)\tag{4}$$

Now we take $s=0$. Let us recall that, for $s=0$, we have a constant speed geodesic, thus we get

$$\left(\sum_{i,j}g_{i,j}(\phi(t,0))\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\right)^{1/2}=d(X_n,X_{n+1}).$$

Thus, due to this and integrating by part in (4), we get

$$\begin{aligned}\frac{1}{2d(X_n,X_{n+1})}\int_0^1\Bigg(&\sum_{i,j}\sum_k\frac{\partial g_{i,j}}{\partial x_k}(\phi(t,0))\frac{\partial\phi_k}{\partial s}\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}-2\sum_{i,j}\frac{\partial^2\phi_i}{\partial t^2}\frac{\partial\phi_j}{\partial s}\\&-2\sum_{i,j}\sum_k\frac{\partial g_{i,j}}{\partial x_k}(\phi(t,0))\frac{\partial\phi_k}{\partial t}\frac{\partial\phi_i}{\partial s}\frac{\partial\phi_j}{\partial t}\Bigg)dt\\&+\frac{1}{d(X_n,X)}\sum_{i,j}g_{i,j}(\phi(t,0))\frac{\partial\phi_i}{\partial s}\frac{\partial\phi_j}{\partial t}.\end{aligned}\tag{5}$$

Now using the fact that $t\mapsto\phi(t,0)$ is a geodesic, the integral term in (5) vanishes.

Thus we have

$$\frac{d}{ds}(s\mapsto d(X_n,\phi_n(1,s)))_{s=0}=\frac{1}{d(X_n,X)}\left\langle\frac{\partial\phi}{\partial t}(0,1),\frac{\partial\phi}{\partial s}(0,1)\right\rangle_g.\tag{6}$$

Finally, the infinitesimal variation associated to (2) with respect to the variation given by $\phi_n$ is given by

$$\frac{1}{\delta t}\left\langle\frac{\partial\phi_n}{\partial t}(0,1),\frac{\partial\phi_n}{\partial s}(0,1)\right\rangle+\left\langle\nabla_g\mathcal{E},\frac{\partial\phi_n}{\partial s}(0,1)\right\rangle=0.$$

The Euler-Lagrange equation associated to (2) is thus :

$$\frac{\frac{\partial\phi_n}{\partial t}(0,1)}{\delta t}+\nabla_g\mathcal{E}(X_{n+1})=0.\tag{7}$$

Let us point out that this condition is natural. Indeed when $M = \mathbb{R}^N$ with the standard euclidean metric, the constant speed geodesic joining $X_n$ to $X_{n+1}$ in a time duration 1 is $t \mapsto X_n + t(X_{n+1} - X_n)$ which gives the standard Euler implicit scheme associated to (1).

Let us give some estimates that will be useful in the sequel.

Since we can assume that $M$ is isometrically embedded in $\mathbb{R}^N$ we deduce that there is a universal constant $A$ such that

$$A||X_{n+1} - X_n|| \leq d(X_n, X_{n+1}). \tag{8}$$

Since $t \mapsto \phi_n(t, 0)$ is a constant speed geodesic joigning $X_n$ to $X_{n+1}$ for $t \in [0, 1]$ we have for some universal constant $B$, according to (8) and $||\frac{\partial \phi_n}{\partial t}(1, 0)||_g = d(X_n, X_{n+1})$,

$$A||X_{n+1} - X_n|| \leq ||\frac{\partial \phi_n}{\partial t}(1, 0)||_g \leq B||X_{n+1} - X_n||. \tag{9}$$

Due to the left hand side of this estimate and by following the proof of the Merlet & Pierre results of [11], we will show that the sequence $(X_n)$ converges to some critical point of $\mathcal{E}$.

## 3. Proof of the main result

In this section we give the proof of our main result, namely theorem (2.2). Let us note first that the semi-convexity and coercivity of $\mathcal{E}$ assumptions are just given to ensure the existence and uniqueness of the sequence $(X_n)$ given $\delta_t$.

We now closely follow [11].

Let us note that since

$$X_{n+1} = \text{argmin} \frac{d(X_n, X)^2}{2\delta_t} + \mathcal{E}(X),$$

we have

$$\frac{d(X_n, X_{n+1})^2}{2\delta_t} + \mathcal{E}(X_{n+1}) \leq \mathcal{E}(X_n). \tag{10}$$

The sequence $(\mathcal{E}(X_n))_{n \in \mathbb{R}^N}$ is therefore non increasing, converges due to our assumptions, and thus there exists a subsequence of $(X_n)$ that converges to some $X_\infty$. Note that we also have $\lim_{n\to\infty} d(X_n, X_{n+1}) = 0$. Note also that according to (7), (9), (10) $\nabla\mathcal{E}(X_\infty) = 0$.

Due to the Lojasiewicz inequality (3), there exist $\nu \in (0, 1/2]$, $\sigma > 0$ and $\gamma > 0$ such that

$$\forall X \in M, \quad d(X, X_\infty) < \sigma \Rightarrow |\mathcal{E}(X)|^{1-\nu} \leq ||\nabla\mathcal{E}(X)||. \tag{11}$$

Let $n$ such that $d(X_{n+1}, X_\infty) < \sigma$. Now, as in Merlet & Pierre [11], two situations are to be treated.

Assume first that $\mathcal{E}(X_{n+1}) > \frac{1}{2}\mathcal{E}(X_n)$.

$$\begin{aligned} \mathcal{E}(X_n)^\nu - \mathcal{E}(X_{n+1})^\nu &= \int_{\mathcal{E}(X_{n+1})}^{\mathcal{E}(X_n)} \nu x^{\nu-1} dx \\ &\geq 2^{\nu-1}\nu\mathcal{E}(X_{n+1})^{\nu-1}|\mathcal{E}(X_n) - \mathcal{E}(X_{n+1})|. \end{aligned}$$

According to (10) and (9), we get, for $C_1 = A^2$

$$\begin{aligned}\mathcal{E}(X_n)^\nu - \mathcal{E}(X_{n+1})^\nu &\geq C_1 2^{\nu-2}\frac{||X_{n+1}-X_n||^2}{\delta t(\mathcal{E}(X_{n+1})^{1-\nu}}\\ &\geq C_1 2^{\nu-2}\nu||X_{n+1}-X_n||\frac{||\nabla\mathcal{E}(X_{n+1})||_g}{\mathcal{E}(X_{n+1})^{1-\nu}}\\ &\geq C_1\frac{2^{\nu-2}\nu}{\gamma}||X_{n+1}-X_n||,\end{aligned}$$

by the Lojasiewicz' inequality (3) .

Assume now that $\mathcal{E}(X_{n+1}) \leq \frac{1}{2}\mathcal{E}(X_n)$.

We have, for $C_2 = \frac{1}{A}$, that

$$\begin{aligned}||X_{n+1}-X_n|| &\leq C_2\sqrt{2\delta t}|\mathcal{E}(X_n)-\mathcal{E}(X_{n+1})|^{1/2}\\ &\leq C_2\sqrt{2\delta t}|\mathcal{E}(X_n)|^{1/2}\\ &\leq C_2(1-\frac{1}{\sqrt{2}})^{-1}\sqrt{2\delta t}(|\mathcal{E}(X_n)|^{1/2}-|\mathcal{E}(X_{n+1})|^{1/2}).\end{aligned}$$

Thus in both cases, we get that for all $n$ such that $d(X_{n+1}, X_\infty) < \sigma$, we have

$$\begin{aligned}||X_{n+1}-X_n|| &\leq \frac{2^{2-\nu}\gamma}{C_1\nu}(|\mathcal{E}(X_n)|^\nu - |\mathcal{E}(X_{n+1})|^\nu)\\ &\quad +5C_2\sqrt{\delta t}(|\mathcal{E}(X_n)|^{1/2}-|\mathcal{E}(X_{n+1})|^{1/2}).\end{aligned}$$

Let $\bar{\mathcal{E}} > 0$ small enough such that

$$\frac{2^{2-\nu}\gamma}{C_1\nu}\bar{\mathcal{E}}^\nu + 5C_2\sqrt{\delta t}\bar{\mathcal{E}}^{1/2} < \sigma/3.$$

Let $\bar{n}$ large enough such that $||X_n - X_\infty|| < \sigma/3$ and $\mathcal{E}(X_n) < \bar{\mathcal{E}}$ and $N$ the largest integer such that $||X_n - X_\infty|| < 2\sigma/3$ for all $n$ such that $\bar{n} \leq n \leq N$. Assume that $N$ is finite. We have

$$\begin{aligned}||X_{N+1}-X_\infty|| &\leq ||X_N - X_\infty|| + ||X_{N+1}-X_N||\\ &\leq ||X_N - X_\infty|| + \sqrt{2\frac{\delta_t}{A^2}\mathcal{E}(X_n)} < \sigma.\end{aligned}$$

Thus we get

$$\sum_{n=\bar{n}}^{N}||X_{n+1}-X_n|| \leq C_1\frac{2^{2-\nu}}{\nu}\mathcal{E}(X_{\bar{n}})^\nu + 5C_2\sqrt{\delta_t}\mathcal{E}(X_{\bar{n}})^{\frac{1}{2}} \leq \frac{\sigma}{3}.$$

This implies

$$||X_{N+1}-X_\infty|| \leq ||X_{\bar{n}} - X_\infty|| + \frac{\sigma}{3} < 2\frac{\sigma}{3},$$

which is a contradiction if $N$ is finite.

As a consequence the sequence $(X_n)$ converges which ends the proof of the main result.

## 4. Some generalizations and partial extensions

In the work by Chill & al [4] , the equation (1) is also considered, as well as the so-called quasi-gradient system on $\mathbb{R}^N$

$$\ddot{x} + \dot{x} + \nabla\mathcal{F}(x) = 0, \tag{12}$$

as a particular case of a more general system on $\mathbb{R}^M$

$$\dot{x} + F(x) = 0. \tag{13}$$

In [2], it is shown that if there exists a continuously differentiable, strict Lyapunov function $\mathcal{E}$ for (13), then there exists a riemannian metric $g$ on the open set $M = \{x \in \mathbb{R}^M,\ F(u) \neq 0\}$ such that $F = \nabla_g \mathcal{E}$.

We will here assume the existence of this function $\mathcal{E}$.

Some fundamental properties of the metric $g$ are strongly related to the so-called compatibility condition (C) and angle condition (AC).

Let us recall the following definitions. The first is given in [4] (see also [2]). This angle condition (AC) has first appeared in [1].

We will say that $\mathcal{E}$ and $F$ satisfies the angle condition (AC) iff there exists $a > 0$ such that

$$\langle \nabla\mathcal{E}, F\rangle \geq a\|\nabla\mathcal{E}\|\|F\|. \tag{14}$$

We will also need the following one, given in [2].
We will say that $\mathcal{E}$ and $F$ satisfies the compatibility condition (C) iff there exist $c_1, c_2 > 0$ such that

$$c_1\|\nabla\mathcal{E}\| \leq \|F\| \leq c_2\|\nabla\mathcal{E}\|. \tag{15}$$

The following result is proven in [2]

**Theorem 4.1.** *The euclidean metric and the metric $g$ are equivalent on $M$ if and only if $\mathcal{E}$ and $F$ satisfy the conditions (AC) and (C).*

Though this property has a very nice appearance, it is not clear that it can be used according to the first section of the present paper. Indeed, in order to do so, one has to check that this metric $g$ can be extended or not to $\mathbb{R}^M$ to a geodesic convex metric. If so, the results of part (1) can be applied.

Otherwise the situation is not clear. In this case we modify the algortihm given in section one the following way.

We will moreover assume that $\mathcal{E}$ is non-negative, that its infimum is 0 and that $\{x,\ F(x) = 0\}$ is compact (for the initial topology). We choose $R > 0$ such that $\{x,\ F(x) = 0\} \subset B_o(0, R)$.

We take $\varepsilon > 0$ such that $\varepsilon < m$. Let $\mathcal{M}_{\varepsilon/2,2R}$ be the manifold $\{x,\ \varepsilon/2 < \mathcal{E}(x),\ \|x\| < 2R\}$ and let $g$ be the metric constructed in [2]. By compactness, it is standard there exists $\rho > 0$ such that for any $x \in \mathcal{M}_{2\varepsilon/3,3R/2}$, the geodesic ball $\mathcal{B}_g(x, \rho) = \exp_x(B(0, \rho))$ is geodesic convex (see [14]).

We can moreover assume that $\rho < 1$ and, moreover, if $x \in \overline{\mathcal{M}_{\varepsilon,R}}$

$$\overline{\mathcal{B}_g(x, \rho)} \in \mathcal{M}_{2\varepsilon/3,3R/2} \tag{16}$$

We choose $x_0 \in \overline{\mathcal{M}_{\varepsilon,R}}$ and consider the following minimization problem:

$$\min\ \mathcal{E}(x) + \frac{d(x_0, x)^2}{2\delta_t}, \quad x \in \overline{\mathcal{B}_g(x_0, \rho)}. \tag{17}$$

By compactness of the ball, the existence of a minimizer is obvious. We denote $x_1$ such a minimizer.
Assume that we have constructed $x_1, ..., x_N$. We have two possibilities.
Either $x_N \in \overline{\mathcal{M}_{\varepsilon,R}}$ so that we take

$$x_{N+1} \in \arg\min_{\overline{\mathcal{B}_g(x_0,\rho)}} \mathcal{E}(x) + \frac{d(x_0,x)^2}{2\delta_t} \tag{18}$$

or $x_N \in \mathcal{M}_{2\varepsilon/3,3R/2} \setminus \overline{\mathcal{M}_{\varepsilon,R}}$ .

We then go on by replacing $\varepsilon$ by $\varepsilon/2$, and this defines the sequence.

Now let us study the convergence of this sequence.

Assume that for some $\varepsilon > 0$ we have

$$\forall n \in \mathbb{N},\ x_n \in \overline{\mathcal{M}_{\varepsilon,R}}. \tag{19}$$

Let $n_k$ an increasing injection of $\mathbb{N}$ such that $(x_{n_k})$ converges and let $l$ denotes the limit. Let $x_{n_k}$ and $x_{n_{k'}}$ such that $l \in \overline{\mathcal{B}_g(x_{n_k},\rho/4)} \cap \overline{\mathcal{B}_g(x_{n_{k'}},\rho/4)}$. This is impossible since this would imply

$$\mathcal{E}(x_{n_k}) < \mathcal{E}(x_{n_{k'}}) < \mathcal{E}(x_{n_k})$$

The same proof implies that either there exists a $N$ such that $\mathcal{E}(x_N) = 0$ or $\forall \varepsilon > 0$ there exists $N$ such that $\forall n > N$

$$0 < \mathcal{E}(x_n) < \varepsilon.$$

Indeed, if there exists $\varepsilon > 0$ such that for any $N$, there is $n_N > N$ such that $x_{n_N} \in \overline{\mathcal{M}_{\varepsilon,R}}$ and we can apply our preceding argument. The only other possibilities are that the sequence $(x_n)$ is stationnary from a certain rank.

This in fact does not imply the convergence of the sequence which is for the moment being unknown to us.

Let us remark that if the metric $g$ is globally defined, then the same argument as the proof of the main theorem shows that the sequence converges if we moreover assume (without loss of generality) that $\rho$ is small enough in order to have for every $N$

$$\min_{\mathcal{S}_g(x_N,\rho)} \mathcal{E} > \frac{1}{2}\mathcal{E}(x_n),$$

where $\mathcal{S}_g(x_N,\rho)$ is the sphere $\overline{\mathcal{B}_g(x_N,\rho)} \setminus \mathcal{B}_g(x_N,\rho)$.

Indeed, in this case, $\rho$ can be chosen globally on the set on the open set $B(0,2R)$ and the same strategy applies as in the proof of the main theorem.

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*Laboratoire M2N, EA7340, CNAM, 292 rue Saint-Martin,, 75003 Paris, France*
*Email address*: thierry.horsin@lecnam.net

*Université de Carthage, Institut Préparatoire aux Etudes Scientifiques et Techniques,, B.P. 51 2070 La Marsa, Tunisia*
*Email address*: ma.jendoubi@fsb.rnu.tn