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# Non genericity of initial data with punctual $\omega$ -limit set

Thierry Horsin and Mohamed Ali Jendoubi

**Abstract.** We show the existence of a nonlinear differential equation of gradient type for which the set of convergence to a point is non generic

## 1. Introduction

Let us consider a function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  of class  $C^\infty$  (the regularity is not our concern for the moment being). It defines a differential equation of gradient type:

$$\begin{cases} x'(t) = -\nabla f(x(t)) \\ x(0) = x_0. \end{cases} \quad (1)$$

Obviously the map  $t \mapsto f(x(t))$  is decreasing for a solution of (1). It is also well known that as we define

$$\omega(x_0) := \bigcap_{t \geq 0} \overline{\{x(s), s \geq t\}}, \quad (2)$$

then each set  $\omega(x_0)$  is connected and compact when  $x$  is bounded. This is the case, for example, when  $f$  is proper.

For years and still now, the structure of the omega-limit set  $\omega(x_0)$  has been widely studied and the properties of its equivalent counterpart for system of infinite dimension is a very active area of interests.

A natural question that arises concerning  $\omega(x_0)$  is whether it is a mere single point (of course an equilibrium point, *i.e.* a *critical point of  $f$* ). In this case we say that the trajectory converges. It is obviously the case if  $\nabla f^{-1}(\{0\})$  is discrete. In two fundamental papers [4] and [5], S. Łojasiewicz has proved that it is also the case when  $f$  is real analytic.

In the seminal book of J. Palis and W. De Melo ([6]), one can find an example of function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for some  $x_0$ ,  $\omega(x_0)$  is not a single point (see also [1]). Of course such a  $f$  cannot be real analytic on the whole space.

In this paper we prove the following: Let  $\mathcal{S}$  be the circle of center 0 and radius 1 in  $\mathbb{R}^2$ .

**Theorem 1.1.** *There exists a  $C^\infty$  function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that there exists an open set  $\mathcal{O}$  satisfying:*

$$\forall x_0 \in \mathcal{O}, \quad \omega(x_0) = \mathcal{S}.$$

**Remark 1.** Let  $\Omega$  be a bounded regular open set of  $\mathbb{R}^N$ . For  $g : \mathbb{R} \rightarrow \mathbb{R}$ , we introduce the semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = g(u), & \text{on } (0, \infty) \times \Omega \\ u = u_0 & \text{on } \{0\} \times \Omega \\ u = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \quad (3)$$

We consider the functional on  $H_0^1(\Omega)$

$$\Gamma(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(u) dx,$$

where  $G$  satisfies  $G' = g$ .

It is standard that (3) can be viewed as a gradient system on  $H_0^1(\Omega)$

$$u' = -\nabla \Gamma(u).$$

Several results about convergence for the equation (1) have been extended to (3) with specific assumptions on  $g$  which we do not detail here.

The example of the book of J. Palis and W. De Melo ([6]) has been used in a penetrating analysis by P. Polacik and K. Rybakowski [7] (see also [8]) to extend the result to a nonlinear heat equation (see (3)).

In [3], P.L. Lions proved that for the semilinear equation, under some natural assumptions on  $g$ , there exists a dense open set of  $u_0$  such that the solution converges to an equilibrium point. Our theorem proves that this is not the case in general for finite dimensional system.

**Remark 2.** In comparison to the example given in [7] whose construction relies on a connexity argument, our method relies on an explicit construction of surrounding curves, playing the role of sub and supersolutions.

**Remark 3.** In [2], it is remarked that the non-convergence of a solution of (1) implies the non-convergence of the solution of the second order differential equation

$$x'' + \alpha x' + \nabla \chi(x) = 0 \quad (4)$$

when one considers  $\chi = \alpha f - \frac{1}{2} |\nabla f|^2$  with the following initial data  $x(0) = x_0$  and  $x'(0) = x_1 = -\nabla f(x_0)$ .

Our result does not extend to the second order equation (4). In fact the set  $\{(x_0, x_1)\}$  that we can construct with our method contains a submanifold of dimension 2 in  $\mathbb{R}^4$ . We skip the proof of this result. The existence of  $\chi$  such that there exist an open set of initial data  $\{(x_0, x_1)\}$  such that the solution to (4) does not converge to an equilibrium point is, to our knowledge, open.

## 2. Proof of the theorem 1.1

### 2.1. Construction of $f$

Let us consider two  $C^\infty$  curves defined in polar coordinate by two functions  $\gamma_i$   $i = 1, 2$  such that

$$\forall \theta \in \mathbb{R}^+, 1 < \gamma_2(\theta + 2\pi) < \gamma_1(\theta) < \gamma_2(\theta) \quad (5)$$

$$\lim_{\theta \rightarrow \infty} \gamma_i(\theta) = 1. \quad (6)$$

Let us also consider an even  $C^\infty$  function

$$F : \mathbb{R} \longrightarrow [0, +\infty[,$$

such that the support of  $F$  is  $[-1, 1]$  and for some  $\alpha \in (0, 1)$

$$F(\alpha) = \max_{[-1, 1]} F,$$

$F$  is strictly increasing on  $[0, \alpha]$  and  $F$  is positive on  $] -1, 1[$ . An example of such function is given by

$$F(x) = \begin{cases} \frac{e^{-\frac{1}{\lambda(1+x)^2} - \frac{1}{\lambda(1-x)^2}}}{(1+x)^\nu(1-x)^\nu}, & \text{when } |x| < 1 \\ 0 & \text{else,} \end{cases}$$

for  $\nu = 1, 2$  and  $\lambda > 0$  large enough with respect to  $\nu$ .

Let us fix  $\theta_0 \in (0, 2\pi)$ . We introduce a function  $\Pi$  such that its support is connected and contains  $\gamma_1(0)$  and  $\gamma_2(0)$  as interior points and such that for  $\theta > \theta_0$   $\gamma_1(\theta)$  and  $\gamma_2(\theta)$  are not in the support of  $\Pi$ .

Let us consider now some point  $M := (x, y) \in \mathbb{R}^2$  such that  $r := \sqrt{x^2 + y^2} > 1$ .

We choose some  $\theta \in [0, 2\pi)$  such that  $M = (r \cos(\theta), r \sin(\theta))$ .

If there does not exist  $p \in \mathbb{N}$  such that  $\gamma_1(\theta + 2p\pi) \leq r \leq \gamma_2(\theta + 2p\pi)$  we define  $\phi(x, y) = 0$ .

Otherwise, there exist a unique  $p \in \mathbb{N}$  such that

$$\gamma_1(\theta + 2p\pi) \leq r \leq \gamma_2(\theta + 2p\pi).$$

In that case we consider

$$\begin{aligned} \phi(x, y) &:= \\ F \left( \frac{r - \gamma_1(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)} + \frac{r - \gamma_2(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)} \right), \end{aligned} \quad (7)$$

and thereafter define  $f$  by

$$f(x, y) := e^{-(\theta + 2p\pi)^2} \phi(x, y) (1 - \Pi(x, y)). \quad (8)$$

From now on, we will choose  $\gamma_1(\theta) = 1 + e^{-\theta}$  and  $\gamma_2(\theta) = 1 + \mu e^{-\theta}$  with  $\mu > 1$ , but we would like to point out that many other choices are possible.

With these functions we will define

$$X(x, y) := \nabla f(x, y).$$

In order to prove our main result we need to check that  $f \in C^\infty(\mathbb{R}^2)$ . According to our choice of  $F$  and our construction, it is clear the  $f \in C^\infty(\Omega_1 \cup \Omega_2)$  where  $\Omega_1 = \{(x, y), x^2 + y^2 < 1\}$  and  $\Omega_2 = {}^c\Omega_1$ .

Let us prove that  $f$  is continuous at any  $(x, y)$  such that  $x^2 + y^2 = 1$

For that we fix such a  $(\bar{x}, \bar{y})$  and a sequence  $(x_n, y_n) \rightarrow (\bar{x}, \bar{y})$ . We may of course assume that  $f(x_n, y_n) \neq 0$ .

Let us assume first that  $(\bar{x}, \bar{y}) = (\cos(\bar{\theta}), \sin(\bar{\theta}))$  for  $\bar{\theta} \in (0, 2\pi)$ .

We can then assume that for some  $\theta_n \in [0, 2\pi)$  we have

$(x_n, y_n) = (\rho_n \cos(\theta_n), \rho_n \sin(\theta_n))$  where  $\rho_n > 1$  and thus up to a subsequence, we have either  $\theta_n \rightarrow \bar{\theta}$  or  $\bar{\theta} + 2\pi$ . Let  $p_n \in \mathbb{N}$  be such that  $\gamma_1(\theta_n + 2p_n\pi) < \rho_n < \gamma_2(\theta_n + 2p_n\pi)$ . We claim that  $p_n \rightarrow \infty$ .

Indeed otherwise, up to a subsequence, we may assume that  $p_n$  is bounded and thus constant for  $n$  large enough and the sequence  $(\theta_n + 2p_n\pi)$  is thus bounded. But this contradicts the fact that  $\rho_n \rightarrow 1$ .

Since  $\theta_n + 2p_n\pi \rightarrow +\infty$ ,  $f(x_n, y_n) \rightarrow 0$ .

We have thus proven the continuity of  $f$ .

Let us prove that  $f$  is  $C^1$ .

Let us denote  $h(x, y) = e^{-(\theta+2p\pi)^2}$ .

We classically define

$$e_r(\theta) = (\cos(\theta), \sin(\theta)), \quad e_\theta(\theta) = (-\sin(\theta), \cos(\theta)).$$

Then  $X = \nabla f$  is given at  $(x, y)$  such that  $f(x, y) \neq 0$ .

$$\begin{aligned} X(x, y) = & -\frac{2h(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)} F'(a(r, \theta + 2p\pi)) e_r \\ & - \frac{h'(\theta + 2p\pi)}{r} F(a(r, \theta + 2p\pi)) e_\theta \\ & - \frac{h(\theta + 2p\pi)}{r} F'(a(r, \theta + 2p\pi)) \times \\ & \frac{1}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)} (-(\gamma_1'(\theta + 2p\pi) + \gamma_2'(\theta + 2p\pi)) \\ & - (2r - \gamma_1(\theta + 2p\pi) - \gamma_2(\theta + 2p\pi)) \frac{\gamma_2'(\theta + 2p\pi) - \gamma_1'(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)}) e_\theta, \end{aligned}$$

where

$$a(r, \theta) := \frac{r - \gamma_1(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)} + \frac{r - \gamma_2(\theta + 2p\pi)}{\gamma_2(\theta + 2p\pi) - \gamma_1(\theta + 2p\pi)},$$

and  $p$  has been chose above.

If we compute  $X(x_n, y_n)$  as before, since we will have  $p_n \rightarrow \infty$  with our choice of  $h$ ,  $X(x_n, y_n)$  tends to 0. Thus the partial derivatives are continuous on  $\mathbb{R}^2$ .

Now a single argument by induction allows us to conclude that  $f \in C^\infty(\mathbb{R}^2)$ .

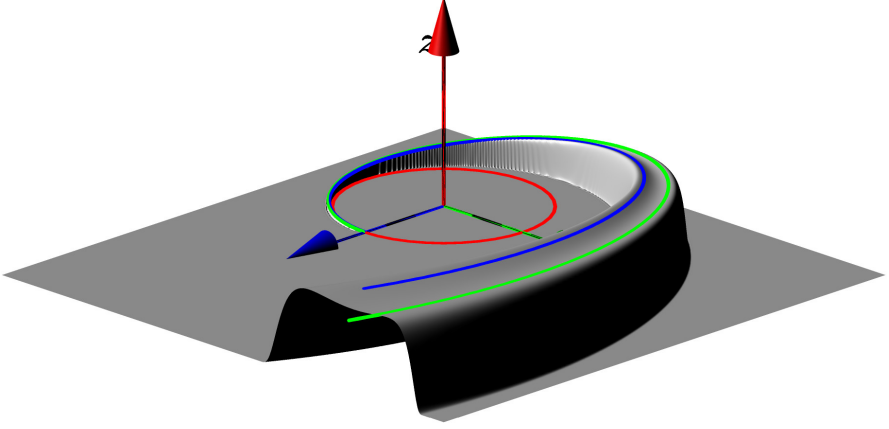


FIGURE 1. profile of  $f$  with  $F(x, y) = \frac{e^{-\frac{1}{3(1+x)^2} - \frac{1}{3(1-x)^2}}}{(1+x)(1-x)}$ . In red  $S$ , and in blue and green the curves  $b_1$  and  $b_2$ .

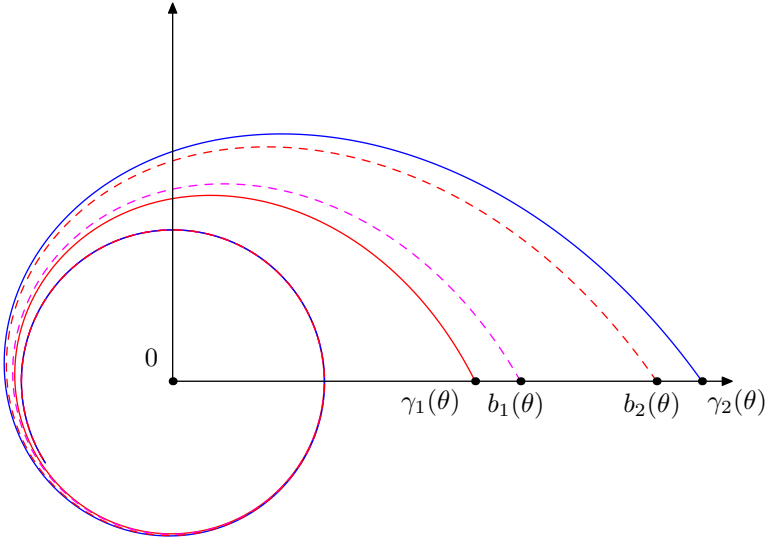


FIGURE 2. The domain on which  $f$  is not 0

## 2.2. $f$ satisfies our problem

In this section we fix some  $(x, y)$  such that  $f(x, y) \neq 0$ . As in the previous section we define  $r$  and  $\theta$  and  $p$ . However in order to simplify the exposition, since we will study the behavior of  $X$  along the half-line  $\theta$  constant, we will replace  $\theta + 2p\pi$  by  $\theta$ , and what we have in mind is described in the pictures [1](#) and [2](#).

There are two values of  $r$ ,  $a_1(\theta)$  and  $a_2(\theta)$ , such for a given  $\theta$   $\phi$  is maximum at  $a_1$  and  $a_2$ . There are also two values of  $r$ ,  $b_1(\theta)$  and  $b_2(\theta)$  such that

$$[b_1(\theta), b_2(\theta)] \subset ]a_1(\theta), a_2(\theta)[,$$

and such that at those values  $\gamma := |\partial_r \phi(r, \theta)|$  is maximum on  $[a_1(\theta), a_2(\theta)]$ . Of course

$$\partial_r \phi(b_1(\theta), \theta) < 0$$

and

$$\partial_r \phi(b_2(\theta), \theta) > 0.$$

These two values are given by

$$b_i(\theta) := \frac{\pm \beta (\gamma_2(\theta) - \gamma_1(\theta))}{2} + \frac{\gamma_1(\theta) + \gamma_2(\theta)}{2}$$

with  $+$  when  $i = 2$  and  $-$  when  $i = 1$  and  $\pm \beta$  are the values in  $[-\alpha, \alpha]$  where  $|F'|$  is maximum on  $[-\alpha, \alpha]$ , while  $F$  is maximum in  $-\alpha$  and  $\alpha$ . We will denote  $\|F'\|_\infty = \max_{[-\alpha, \alpha]} |F'|$ .

One can easily compute that

$$b_2(\theta) = \gamma_1(\theta)(1 - \frac{\beta}{2}) + \gamma_2(\theta)(1 + \frac{\beta}{2}),$$

and

$$b_1(\theta) = \gamma_1(\theta)(1 + \frac{\beta}{2}) + \gamma_2(\theta)(1 - \frac{\beta}{2}),$$

and since  $\beta \in ]-1, 1[$ , the  $b_i$ s are decreasing.

The tangent vector  $\tau_i(\theta)$  to the curve  $b_i(\theta)$  is given by

$$b'_i(\theta)e_r + b_i(\theta)e_\theta$$

The normal vector is then  $n_i(\theta) = -b_i(\theta)e_r + b'_i(\theta)e_\theta$ .

In order to prove our result, we show that for  $\theta$  large enough one has

$$X(b_1(\theta), \theta) \cdot n_1(\theta) < 0 \text{ and } X(b_2(\theta), \theta) \cdot n_2(\theta) > 0.$$

Let us recall that

$$\begin{aligned} X(r, \theta) &= -\frac{2h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} F'(a(r, \theta))e_r - \frac{h'(\theta)}{r} F(a(r, \theta))e_\theta \\ &\quad - \frac{h(\theta)}{r} F'(a(r, \theta)) \frac{1}{\gamma_2(\theta) - \gamma_1(\theta)} (-(\gamma'_1(\theta) + \gamma'_2(\theta)) \\ &\quad - (2r - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma'_2(\theta) - \gamma'_1(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}) e_\theta. \end{aligned}$$

Thus

$$\begin{aligned} X(r, \theta) \cdot n_i(\theta) &= 2 \frac{h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} F'(a(r, \theta)) b_i(\theta) \\ &\quad - \frac{h'(\theta)}{r} F(a(r, \theta)) b'_i(\theta) - b'_i(\theta) \frac{h(\theta)}{r} \frac{F'(a(r, \theta))}{\gamma_2(\theta) - \gamma_1(\theta)} [-(\gamma'_1(\theta) + \gamma'_2(\theta)) \\ &\quad - (2r - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma'_2(\theta) - \gamma'_1(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}] \end{aligned}$$

Now we take  $r = b_1(\theta)$  and  $i = 1$ .

The preceding computation gives:

$$\begin{aligned} X(b_1(\theta), \theta).n_1(\theta) &= -2 \frac{h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} \|F'\|_\infty b_1(\theta) \\ &\quad - \frac{h'(\theta)}{b_1(\theta)} F(a(b_1(\theta), \theta)) b_1'(\theta) \\ &\quad + b_1'(\theta) \frac{h(\theta)}{b_1(\theta)} \frac{\|F'\|_\infty}{\gamma_2(\theta) - \gamma_1(\theta)} [-(\gamma_1'(\theta) + \gamma_2'(\theta)) \\ &\quad - (2b_1(\theta) - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma_2'(\theta) - \gamma_1'(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}]. \end{aligned}$$

Inside the bracket the quantity is

$$-(\gamma_1'(\theta) + \gamma_2'(\theta)) + \beta(\gamma_2'(\theta) - \gamma_1'(\theta))$$

Let us recall that  $\gamma_1(\theta) = 1 + e^{-\theta}$  and  $\gamma_2(\theta) = 1 + \mu e^{-\theta}$ , we thus get

$$X(b_1(\theta), \theta).n_1 = \tag{9}$$

$$-2 \frac{h(\theta)}{(\mu - 1)e^{-\theta}} \|F'\|_\infty (1 + (\frac{1 + \mu}{2} + \beta(\frac{1 - \mu}{2}))e^{-\theta}) \tag{10}$$

$$+ h'(\theta) F(a(b_1(\theta), \theta)) \frac{\frac{\mu+1}{2} + \beta \frac{1-\mu}{2}}{1 + e^{-\theta}(\frac{\mu+1}{2} + \beta \frac{1-\mu}{2})} e^{-\theta} \tag{11}$$

$$- \frac{h(\theta) \|F'\|_\infty (\frac{\mu+1}{2} + \beta \frac{1-\mu}{2})}{(1 + e^{-\theta}(\frac{\mu+1}{2} + \beta \frac{1-\mu}{2}))(\mu - 1)} ((1 + \beta) + \mu(1 - \beta)) e^{-\theta} \tag{12}$$

while

$$\begin{aligned} X(b_2(\theta), \theta).n_2(\theta) &= 2 \frac{h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} \|F'\|_\infty b_2(\theta) \\ &\quad - \frac{h'(\theta)}{b_2(\theta)} F(a(r, \theta), \theta) b_2'(\theta) \\ &\quad - b_2'(\theta) \frac{h(\theta)}{b_2(\theta)} \frac{\|F'\|_\infty}{\gamma_2(\theta) - \gamma_1(\theta)} [-(\gamma_1'(\theta) + \gamma_2'(\theta)) \\ &\quad - (2b_2(\theta) - \gamma_1(\theta) - \gamma_2(\theta)) \frac{\gamma_2'(\theta) - \gamma_1'(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)}] \end{aligned}$$

which, with the previous choice of  $\gamma_i$  leads to

$$\begin{aligned} X(b_2(\theta), \theta).n_2(\theta) &= 2 \frac{h(\theta)}{\gamma_2(\theta) - \gamma_1(\theta)} \|F'\|_\infty b_2(\theta) \\ &\quad - \frac{h'(\theta)}{b_2(\theta)} F(a(r, \theta), \theta) b_2'(\theta) \\ &\quad - b_2'(\theta) \frac{h(\theta)}{b_2(\theta)} \frac{\|F'\|_\infty}{\gamma_2(\theta) - \gamma_1(\theta)} [(1 + \mu) + \beta(\mu - 1)] e^{-\theta} \end{aligned}$$



Let us remark that in the latter equation the second term of the right hand side is neglectable with respect to the first as  $\theta$  goes to infinity, while the third and the first are positive. Thus for  $\theta$  large enough  $X(b_2(\theta), \theta).n_2(\theta) > 0$ .

In the former (the derivation of  $X(b_1(\theta), \theta)$ ) the second and the third part of the right hand side is neglectable with respect to the first which is negative. Thus, if  $\theta$  is large enough  $X(b_1(\theta), \theta).n_1(\theta) < 0$ .

We fix  $\theta_0$  so that for  $\theta > \theta_0$

$$X(b_1(\theta), \theta).n_1(\theta) < 0 \text{ and } X(b_2(\theta), \theta).n_2(\theta) > 0.$$

We now take  $\theta_1 > \theta_0$  so that  $\sup f$  on  $[a_1(\theta_1), a_2(\theta_1)] \cap \theta = \theta_1$  is less than  $\inf f$  on  $[a_1(\theta_0), a_2(\theta_0)] \cap \theta = \theta_2$ . This is always possible due to our choice of  $h$ .

The two curves  $b_1$  and  $b_2$  are such that if  $x_0 \in [b_1(\theta_1), b_2(\theta_1)]$  the solution  $x' = \nabla X(x)$  which is  $x_0$  at  $t = 0$  does not cross  $b_1$  nor  $b_2$ .

The function  $h$  having no critical point, we have proven our result.

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