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## Fault Detection and Isolation for Linear discrete-time Delayed Systems based on $\mathcal{L}_\infty$ Observer and Zonotopic Analysis

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**Abstract:** This paper considers actuator fault detection and isolation (FDI) problem for a class of linear discrete-time systems with constant time-delay. Based on delay-free representation, a novel fault detection and isolation scheme based on a bank of  $\mathcal{L}_\infty$  fault detection observers (FDOs) is proposed. Under the assumption that the disturbances and measurement noise are unknown but bounded,  $\mathcal{L}_\infty$  performance is introduced to reduce the effects of disturbances. Meanwhile, the FDOs are designed to be sensitive to the fault by means of fault detectability indices and matrix. The design conditions of the proposed FDOs are derived in terms of Linear Matrix Inequalities (LMIs). Furthermore, a zonotopic analysis method is presented to achieve the residual evaluation. Simulation results are provided to demonstrate the performance of the proposed approach.

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**Keywords:** Fault detection and isolation, Time-delay systems, Zonotopic analysis,  $\mathcal{L}_\infty$  design

### 1. INTRODUCTION

Time-delay systems have been widely studied in the last decades due to their frequent presence in engineering applications such as chemical and biological processes, hydraulic systems and manufacturing processes (Niculescu and Gu, 2012). For instance, the emergence of delays in dynamical systems was diagnosed as one of the major source of instability and poor performance (Sipahi et al., 2011). Among the various reported problems encountering time-delay systems, Fault Detection and Isolation (FDI) has attracted the interest of many researchers. For instance, ensuring a high level of safety and reliability is a fundamental issue. FDI for time-delay systems have been studied through various methods such as parity space approaches (Wang et al., 2010; Fang et al., 2007), geometric approaches (Qiao et al., 2016; Meskin and Khorasani, 2009) and observer based techniques (Hamdi et al., 2018; Atitallah et al., 2017). From a practical point of view, the presence of model uncertainties, unknown disturbances and noises can affect the performance of fault detection and isolation. Accordingly, robust FDI approaches are required. Many researchers have focused in the field of robust fault diagnosis by introducing  $H_\infty$  performances to analyse the robustness against the disturbances (Bai et al., 2007; Bedjaoui et al., 2006). Note that the  $H_\infty$  norm is poorly adapted to residual evaluation since it is a measurement of energy-to-energy gain while the energies of various practical disturbances or noise signals are unbounded. Instead,  $\mathcal{L}_\infty$  crite-

ri- on, which describes the peak-to-peak performance index, can be considered as an alternative solution to enhance fault detection performances under a more feasible assumption that the disturbances are unknown but bounded (Tabarraie, 2013). For instance, Wang et al. (2017b) proposed the concept of  $H_\infty/\mathcal{L}_\infty$  fault detection observer (FDO) for linear parameter varying systems with unknown disturbance and actuator fault. In (Han et al., 2018),  $H_\infty/\mathcal{L}_\infty$  FDO was extended to linear discrete-time descriptor systems. To the best of the authors' knowledge,  $\mathcal{L}_\infty$  FDOs have not been fully investigated in the case of time-delay systems.

On the other hand, set-membership approaches have been considered as an interesting alternative to deal with the effects of bounded uncertainties which are supposed to belong to compact sets such as intervals, ellipsoids, polytopes and zonotopes. In recent years, fault diagnosis based on zonotopic techniques has received a great attention (Zhu et al., 2021; Tang et al., 2020; Wang et al., 2017a). In the aforementioned references, the zonotope-based FDI approaches have been proven to provide systematic dynamic thresholds for residual evaluation.

By combining the benefits of  $\mathcal{L}_\infty$  FDO design and zonotopic residual evaluation, this paper proposes a novel actuator FDI technique for linear discrete-time systems with constant time-delay. The main contributions lie in the following aspects. Based on the delay-free augmented system equivalent representation, a bank of robust FDOs is designed, based on  $\mathcal{L}_\infty$  performance and fault detectability indices and matrix proposed

in the work of Keller (1999). Each FDO is decoupled from a certain actuator fault while sensitive to the others. Based on these FDOs, residual evaluation is achieved by using zonotopic analysis.

This paper is arranged as follows. In section 2, notations and some preliminaries are presented. Section 3 is devoted to the fault detection and isolation proposed technique. Simulation results are shown in section 4. Section 5 concludes the paper.

## 2. PRELIMINARIES

**Notations.** In the sequel, the following notations are used.  $\mathbb{R}^n$  and  $\mathbb{R}^{m \times n}$  denote respectively the  $n$  and  $m \times n$  dimensional Euclidean spaces.  $\mathcal{C}_h = \mathcal{C}([-h, 0], \mathbb{R})$  represents the set functions defined on the interval  $[-h, 0]$ . The symbol  $I_n$  denotes an identity matrix with dimension  $n$ . For a matrix  $P \in \mathbb{R}^{m \times n}$ ,  $P^T$  and  $P^\dagger$  are used to represent the transpose and the Moore-Penrose pseudo-inverse of  $P$ . If  $P$  is positive (negative) definite, we denote it as  $P \succ 0$  ( $P \prec 0$ ). For a signal  $x_k$ , its  $L_\infty$ -norm is defined as  $\|x\|_\infty = \sup_{k \geq 0} \|x_k\|$ , where  $\|x\| = \sqrt{x_k^T x_k}$ . An asterisk  $*$  represents a term induced by symmetry in a symmetric block matrix. The Minkowski sum is denoted by  $\oplus$ .

The following definitions, property and lemma are used in this paper.

**Definition 1.** (Combastel, 2015) An  $s$ -zonotope  $\mathcal{Z} \subseteq \mathbb{R}^n$  is the affine image of a hypercube  $\mathbb{B}^s = [-1, 1]^s$  in  $\mathbb{R}^n$  and can be expressed as:

$$\mathcal{Z} = \langle p, H \rangle = p \oplus H\mathbb{B}^s, \quad (1)$$

where  $p \in \mathbb{R}^n$  is the center of  $\mathcal{Z}$  and  $H \in \mathbb{R}^{n \times s}$  denotes its generator matrix.

**Definition 2.** (Combastel, 2015) For a zonotope  $\mathcal{Z} \subset \mathbb{R}^n$ , its interval hull  $\text{Box}(\mathcal{Z})$  is the smallest interval vector containing it, which is denoted by:

$$\mathcal{Z} \subseteq \text{Box}(\mathcal{Z}) = [\underline{z}, \bar{z}], \quad (2)$$

where  $[\underline{z}, \bar{z}] = \{z \in \mathcal{Z}, \underline{z} \leq z \leq \bar{z}\}$  is an interval vector,  $\underline{z}$  and  $\bar{z}$  are the lower and upper bounds of  $z$ .

**Property 1.** (Combastel, 2015) For zonotopes, the following properties hold:

$$\langle p_1, H_1 \rangle \oplus \langle p_2, H_2 \rangle = \langle p_1 + p_2, [H_1 \ H_2] \rangle, \quad (3a)$$

$$\langle p, H \rangle \subseteq \langle p, \downarrow_q(H) \rangle \subseteq \langle p, rs(H) \rangle, \quad (3b)$$

where  $p, p_1, p_2 \in \mathbb{R}^n$ ,  $H, H_1, H_2 \in \mathbb{R}^{n \times m}$  and  $K \in \mathbb{R}^{k \times n}$ .

$\downarrow_q(\cdot)$  denotes the reduction operator where  $q$  is the maximum number of columns of the generator matrix after reduction and  $rs(H) \in \mathbb{R}^{n \times n}$  is a row sum diagonal matrix given by:

$$rs(H) = \text{diag}\left\{ \sum_{j=1}^m |H_{1,j}| \cdots \sum_{j=1}^m |H_{n,j}| \right\}.$$

Note that  $\langle p, rs(H) \rangle$  is a box, i.e., an interval vector, which represents a box outer bound of  $\langle p, H \rangle$ .

**Lemma 1.** (Rao et al., 1972) Given matrices  $A \in \mathbb{R}^{a \times b}$ ,  $X \in \mathbb{R}^{b \times c}$  and  $B \in \mathbb{R}^{a \times c}$ . If  $\text{rank}(X) = c$ , then, the general solution of  $AX = B$  is

$$A = BX^\dagger + S(I - XX^\dagger), \quad (4)$$

where  $S \in \mathbb{R}^{a \times b}$  is an arbitrary matrix.

## 3. FAULT DETECTION AND ISOLATION FOR LINEAR DISCRETE-TIME DELAYED SYSTEMS

In this section, a fault detection and isolation (FDI) methodology is developed for linear discrete-time systems with constant

time-delay. The proposed technique consists in the use of a bank of  $L_\infty$  observers, which are robust against disturbances and noise, meanwhile sensitive to fault. The number of observers constituting the observers bank is equal to the number of faults  $n_f$  under consideration. For the designed observers, the residual evaluation and FDI decision logic are achieved based on zonotopic analysis. Fig. 1 shows the schematic of the proposed FDI strategy.

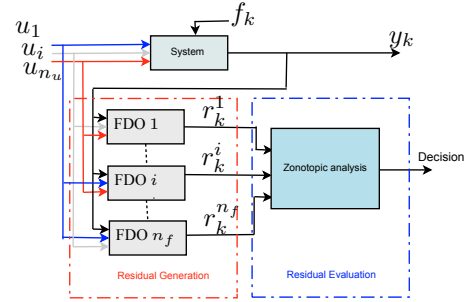


Fig. 1. Schematic of the proposed FDI strategy

Consider the following linear discrete-time delayed system with actuator faults described by:

$$\begin{cases} x_{k+1} = Ax_k + A_h x_{k-h} + Bu_k + D_1 w_k + F f_k, \\ y_k = Cx_k + D_2 v_k, \\ x_k = \phi_k, \quad k \in [-h, 0] \end{cases} \quad (5)$$

where  $x_k \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_u}$ ,  $y_k \in \mathbb{R}^{n_y}$  and  $f_k \in \mathbb{R}^{n_f}$  are the vectors of state, input, measurement output and actuator fault, respectively.  $A$ ,  $A_h$ ,  $B$ ,  $C$ ,  $D_1$  and  $D_2$  are constant matrices of appropriate dimensions.  $F = [F_1 \ F_2 \ \cdots \ F_{n_f}]$  is the fault distribution matrix. It is composed of the columns of  $B$  which correspond to the faulty actuators ( $n_f = n_u$ ).  $w_k \in \mathbb{R}^{n_w}$  is the unknown process disturbances vector and  $v_k \in \mathbb{R}^{n_v}$  is the measurement noise vector.  $\phi_k \in \mathcal{C}_h$  is the initial condition and  $h \in \mathbb{Z}_+$  denotes the constant time-delay.

In this manuscript, the following assumptions are used.

*Assumption 1.* The pair  $(A, C)$  is observable.

*Assumption 2.* The initial state vector  $x_k$ , the process disturbance vector and the measurement noise vector are assumed to be bounded by the following zonotopes:

$$x_k \in \langle p_k, H_k \rangle, \quad w_k \in \mathcal{W} = \langle 0, H_w \rangle, \quad v_k \in \mathcal{V} = \langle 0, H_v \rangle,$$

where  $p_k \in \mathbb{R}^{n_x}$ ,  $H_k \in \mathbb{R}^{n_x \times n_x}$ ,  $H_w \in \mathbb{R}^{n_w \times n_w}$  and  $H_v \in \mathbb{R}^{n_v \times n_v}$  are known vector and matrices.

In order to propose a robust FDI technique for the system (5), we define  $\tilde{x}_k = [x_k^T \ x_{k-h}^T]^T$ . Then, the system (5) may be rewritten as a delay-free augmented system described by:

$$\begin{cases} \tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k + \tilde{D}_1 w_k + \tilde{F}f_k, \\ y_k = \tilde{C}\tilde{x}_k + D_2 v_k, \\ \tilde{x}_k = [\phi_k, 0], \quad k \in [-h, 0] \end{cases} \quad (6)$$

where

$$\tilde{A} = \begin{bmatrix} A & A_h \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \\ \tilde{C} = [C \ 0].$$

Note that the FDI problem for (5) is converted to design a FDI approach for the delay-free augmented system (6).

*Remark 1.* It is clear that the augmented system (6) does not contain any time-delay associated with the state vector  $\tilde{x}_k$ . As a result, the problem of FDO observers design for the system

(6) can be easily solved by employing classical results. To end this note, the pair  $(\tilde{A}, \tilde{C})$  is assumed to be observable at least detectable.

### 3.1 Fault detection observer design

The  $i$ th observer is designed to be decoupled from the  $i$ th actuator fault. The structure is formulated as follows:

$$\begin{cases} \hat{x}_{k+1}^i = \tilde{A}\hat{x}_k^i + \tilde{B}u_k + L_i(y_k - \tilde{C}\hat{x}_k^i), \\ r_k^i = y_k - \tilde{C}\hat{x}_k^i, \quad i = \overline{1, \dots, n_f} \end{cases} \quad (7)$$

where  $\hat{x}_k^i$  is the estimation of  $\tilde{x}_k$ .  $r_k^i$  represents the generated residual signal.  $L_i \in \mathbb{R}^{2n_x \times n_y}$  is the observer gain matrices to be designed.

The objective is to compute the observer gain matrices  $L_i$  to attenuate the effects of the uncertainties and improve the fault sensitivity on residual signal. For this purpose, the definitions of fault detectability indices and fault detectability matrix of system (6) are introduced as follows.

**Definition 3.** (Fault detectability indices (Keller, 1999))

The fault detectability indices of system (6) are defined as  $\rho = \{\rho_1, \rho_2, \dots, \rho_{n_f}\}$  where

$$\rho_i = \min\{v : C\tilde{A}^{v-1}\tilde{F}_i \neq 0, \quad i = \{1, 2, \dots\}\} \quad (8)$$

where  $\tilde{F}_i$  is the  $i$ th column of the matrix  $\tilde{F}$ .

**Definition 4.** (Fault detectability matrix (Keller, 1999))

With the fault detectability indices of system (6) are defined as  $\rho = \{\rho_1, \rho_2, \dots, \rho_{n_f}\}$ , the fault detectability matrix is given by:

$$\Psi = [\tilde{A}^{\rho_1-1}\tilde{F}_1 \quad \tilde{A}^{\rho_2-1}\tilde{F}_2 \quad \dots \quad \tilde{A}^{\rho_{n_f}-1}\tilde{F}_{n_f}]. \quad (9)$$

By defining the estimation error vector  $e_k^i = \tilde{x}_k - \hat{x}_k^i$ , the error dynamics can be formulated as:

$$\begin{cases} e_{k+1}^i = (\tilde{A} - L_i\tilde{C})e_k^i + E_1d_k + \tilde{F}_if_k, \\ r_k^i = \tilde{C}e_k^i + D_2v_k, \end{cases} \quad (10)$$

where  $E_1 = [\tilde{D}_1 \quad -L_iD_2]$ ,  $d_k = [w_k^T \quad v_k^T]^T$ .

For efficient fault detection and isolation, it is essential to attenuate the effects of disturbances and noises and to improve the sensitivity to fault on the residual signals. Herein, the error dynamics (10) can be split into two subsystems where the subsystem (11) is decoupled from the effects of faults, while the subsystem (12) is only affected by the actuator faults.

$$\begin{cases} e_{d,k+1}^i = (\tilde{A} - L_i\tilde{C})e_{d,k}^i + E_1d_k, \\ r_{d,k}^i = \tilde{C}e_{d,k}^i + D_2v_k, \end{cases} \quad (11)$$

$$\begin{cases} e_{f,k+1}^i = (\tilde{A} - L_i\tilde{C})e_{f,k}^i + \tilde{F}_if_k, \\ r_{f,k}^i = \tilde{C}e_{f,k}^i, \end{cases} \quad (12)$$

where  $e_k^i = e_{d,k}^i + e_{f,k}^i$  and  $r_k^i = r_{d,k}^i + r_{f,k}^i$ . For  $k \in [-h, 0]$ ,  $e_{d,k}^i = e_k^i$  and  $e_{f,k}^i = 0$ .

Motivated by the works (Keller, 1999), the analysis of the effects of occurred actuator faults and uncertainties on the residual signals is achieved using the fault detectability indices and matrix given in Definitions 3 and 4. This result is summarized in the following theorem.

**Theorem 1.** Consider the delay-free augmented system (6). If there exists an observer gain matrix  $L_i \in \mathbb{R}^{2n_x \times n_y}$  satisfying:

$$(\tilde{A} - L_i\tilde{C})\Psi_i = 0, \quad (13)$$

where  $\Psi_i = \tilde{A}^{\rho_i-1}\tilde{F}_i$  for  $i = \overline{1, \dots, n_f}$ , then, the effect of the fault on the residual signal  $r_{f,k}^i$  can be expressed as follows:

$$r_{f,k}^i = \Lambda_i f_{i,k-\rho_i}, \quad k > k_f + s, \quad i = \overline{1, \dots, n_f} \quad (14)$$

where  $\Lambda_i = \tilde{C}\Psi_i$ ,  $k_f$  denotes of the time instant of fault occurrence and  $s = \max\{\rho_1, \rho_2, \dots, \rho_{n_f}\}$  represents the maximum value of fault detectability indices.

**Proof.** From (12), we can derive that:

$$\begin{aligned} r_{f,k}^i &= \tilde{C}(\tilde{A} - L_i\tilde{C})e_{f,k-1}^i + \tilde{C}\tilde{F}_if_{k-1} \\ &= \tilde{C}(\tilde{A} - L_i\tilde{C})^2e_{f,k-2}^i + \tilde{C}(\tilde{A} - L_i\tilde{C})\tilde{F}_if_{k-2} \\ &\quad + \tilde{C}\tilde{F}_if_{k-1} \\ &\vdots \\ &= \tilde{C}(\tilde{A} - L_i\tilde{C})^ke_{f,0}^i + \tilde{C}(\tilde{A} - L_i\tilde{C})^{k-k_f}\tilde{F}_if_{k_f} + \dots + \\ &\quad + \tilde{C}(\tilde{A} - L_i\tilde{C})^{k-s}\tilde{F}_if_s + \dots + \tilde{C}(\tilde{A} - L_i\tilde{C})\tilde{F}_if_{k-1}. \end{aligned} \quad (15)$$

where  $k > s + k_f$ .

By means of fault detectability indices given by Definition 3., we have:

$$\tilde{C}(\tilde{A} - L_i\tilde{C})^j\tilde{F}_i = \tilde{C}\tilde{A}^{j-1}\tilde{F}_i = 0, \quad j < \rho_i \quad (16)$$

$$\tilde{C}(\tilde{A} - L_i\tilde{C})^j\tilde{F}_i = \tilde{C}\tilde{A}^{j-1}\tilde{F}_i, \quad j = \rho_i \quad (17)$$

$$\tilde{C}(\tilde{A} - L_i\tilde{C})^j\tilde{F}_i = \tilde{C}(\tilde{A} - L_i\tilde{C})^{j-\rho_i}\tilde{A}^{\rho_i-1}\tilde{F}_i = 0, \quad j > \rho_i \quad (18)$$

By using (13), we obtain:

$$(\tilde{A} - L_i\tilde{C})\tilde{A}^{\rho_i-1}\tilde{F}_i = 0. \quad (19)$$

Substituting (19) into (18) yields:

$$\tilde{C}(\tilde{A} - L_i\tilde{C})^j\tilde{F}_i = 0, \quad j > \rho_i \quad (20)$$

It follows that for all  $j = \{1, \dots, k - k_f\}$ :

$$\tilde{C}(\tilde{A} - L_i\tilde{C})^j\tilde{F}_i = \begin{cases} \tilde{C}\tilde{A}^{j-1}\tilde{F}_i, & j = \rho_i \\ 0, & j \neq \rho_i \end{cases} \quad (21)$$

Since the  $i$ th observer is decoupled from the  $i$ th actuator fault, we have:

$$\begin{aligned} r_f(k) &= \tilde{C}(\tilde{A} - L_i\tilde{C})^ke_{f,0}^i + \tilde{C}\tilde{A}^{\rho_i-1}\tilde{F}_if_{i,k-\rho_i} \\ &= \tilde{C}(\tilde{A} - L_i\tilde{C})^ke_{f,0}^i + \tilde{C}\Psi_if_{i,k-\rho_i}. \end{aligned} \quad (22)$$

Since  $e_{f,0}^i = 0$ , the relation (22) is equivalent to (14) which completes the proof.

In order to measure disturbance attenuation ability, a  $\mathcal{L}_\infty$  criterion is introduced to attenuate state disturbances and measurement noises. The objective is to design  $L_i$  for observer (7) such that the residual satisfies the following  $\mathcal{L}_\infty$  performance for given scalars  $0 < \alpha < 1$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$ :

$$\|r_k^i\| < \sqrt{\gamma_1^2\alpha^kV_0 + \gamma_1^2\|d\|_\infty^2 + \gamma_2^2\|v\|_\infty^2} \quad (23)$$

where  $V_0 = e_{d,0}^i{}^T P_i e_{d,0}^i$  and  $P_i$  is a positive definite matrix to be designed.

To this end, the following theorem is proposed.

**Theorem 2.** Given scalars  $0 < \alpha < 1$ ,  $\gamma_1 > 0$  and  $\gamma_2 > 0$ . If  $\text{rank}(\Lambda_i) = 1$  and if there exist a positive definite matrix  $P_i \in \mathbb{R}^{2n_x \times 2n_x}$  and matrices  $G_i \in \mathbb{R}^{2n_x \times 2n_x}$  and  $Y_i \in \mathbb{R}^{2n_x \times (2n_x + n_y)}$  such that the following LMIs

$$\begin{bmatrix} -\alpha P_i & * & * & * \\ 0 & -(1-\alpha)I_{n_w} & * & * \\ 0 & 0 & -(1-\alpha)I_{n_v} & * \\ \Omega_1 & \Omega_2 & \Omega_3 & P_i - G_i - G_i^T \end{bmatrix} \prec 0, \quad (24)$$

$$\begin{bmatrix} \tilde{C}^T \tilde{C} - \gamma_1^2 P_i & * \\ D_2^T \tilde{C} & D_2^T D_2 - \gamma_1^2 I_{n_v} \end{bmatrix} \prec 0, \quad (25)$$

hold, where

$$\begin{cases} \Omega_1 = G_i \tilde{A} - G_i \tilde{A} \Psi_i \Lambda_i^\dagger \tilde{C} - Y_i (I - \Lambda_i \Lambda_i^\dagger) \tilde{C} \\ \Omega_2 = G_i \tilde{A} \Psi_i \Lambda_i^\dagger \tilde{D}_1 - Y_i (I - \Lambda_i \Lambda_i^\dagger) \tilde{D}_1 \\ \Omega_3 = -G_i \tilde{A} \Psi_i \Lambda_i^\dagger D_2 + Y_i (I - \Lambda_i \Lambda_i^\dagger) D_2 \end{cases} \quad (26)$$

then, the error system (10) is stable and satisfies the performance (23). If the LMIs (24) and (25) are solvable, then the observer gain matrices  $L_i$  are given by:

$$L_i = \tilde{A} \Psi_i \Lambda_i^\dagger G_i^{-1} Y_i (I - \Lambda_i \Lambda_i^\dagger), \quad i = \overline{1, \dots, n_f}. \quad (27)$$

**Proof.** Since  $\text{rank}(\Lambda_i) = 1$ , based on Lemma 1., the observer gain matrices  $L_i$  can be obtained by solving the equation (13). It is given by:

$$L_i = \tilde{A} \Psi_i \Lambda_i^\dagger + G_i^{-1} Y_i (I - \Lambda_i \Lambda_i^\dagger), \quad i = \overline{1, \dots, n_f}. \quad (28)$$

From (26) and (27), the inequality (24) is equivalent to:

$$\begin{bmatrix} -\alpha P_i & * & * \\ 0 & -(1-\alpha)I_{n_w+n_v} & * \\ G_i(\tilde{A} - L_i \tilde{C}) & G_i E_1 & P_i - G_i - G_i^T \end{bmatrix} \prec 0. \quad (29)$$

By pre-multiplying and post-multiplying the inequality (29) with

$$\begin{bmatrix} I_{n_x} & 0 & (\tilde{A} - L_i \tilde{C})^T \\ 0 & I_{n_w+n_v} & E_1^T \end{bmatrix}$$

and its transpose, we obtain:

$$\begin{bmatrix} \omega_1 & \omega_2^T \\ \omega_2 & \omega_3 \end{bmatrix} \prec 0, \quad (30)$$

where

$$\begin{cases} \omega_1 = -\alpha P_i + (\tilde{A} - L_i \tilde{C})^T P_i (\tilde{A} - L_i \tilde{C}) \\ \omega_2 = E_1^T P_i (\tilde{A} - L_i \tilde{C}) \\ \omega_3 = -(1-\alpha)I_{n_w+n_v} + E_1^T P_i E_1 \end{cases}$$

Then, pre-multiply and post-multiply the inequality (30) with  $[e_{d,k}^i \ d_k^T]^T$  and its transpose, respectively, it follows that:

$$\begin{aligned} & -\alpha e_{d,k}^i{}^T P_i e_{d,k}^i + e_{d,k}^i{}^T (\tilde{A} - L_i \tilde{C})^T P_i (\tilde{A} - L_i \tilde{C}) e_{d,k}^i \\ & + e_{d,k}^i{}^T (\tilde{A} - L_i \tilde{C})^T P_i E_1 d_k + d_k^T E_1^T P_i (\tilde{A} - L_i \tilde{C}) e_{d,k}^i \\ & - (1-\alpha) d_k^T d_k + d_k^T E_1^T E_1 d_k \leq 0. \end{aligned} \quad (31)$$

Define the following Lyapunov function:

$$V_k = e_{d,k}^i{}^T P_i e_{d,k}^i, \quad (32)$$

then the relation (31) is equivalent to:

$$V_{k+1} \leq \alpha V_k + (1-\alpha) d_k^T d_k, \quad (33)$$

it follows that:

$$\begin{aligned} V_k & \leq \alpha^k V_0 + (1-\alpha) \sum_{j=0}^{k-1} \alpha^j d_{k-1-j}^T d_{k-1-j} \\ & \leq \alpha^k V_0 + (1-\alpha^k) \|d\|_\infty^2 \\ & < \alpha^k V_0 + \|d\|_\infty^2. \end{aligned} \quad (34)$$

In addition, by pre-multiplying and post-multiplying the inequality (25) with  $[e_{d,k}^i \ v_k^T]^T$ , and its transpose, we have:

$$\begin{aligned} & e_{d,k}^i{}^T \tilde{C}^T \tilde{C} e_{d,k}^i - \gamma_1^2 e_{d,k}^i{}^T P_i e_{d,k}^i + e_{d,k}^i{}^T \tilde{C}^T D_1 v_k \\ & + v_k^T D_2^T \tilde{C} e_{d,k}^i + v_k^T D_2^T D_2 v_k - \gamma_2^2 v_k^T v_k \leq 0. \end{aligned} \quad (35)$$

Since  $r_{d,k}^i = \tilde{C} e_{d,k}^i + D_2 v_k$ , it follows from the relation (35) that:

$$\begin{aligned} \|r_d^i\|^2 & \leq \gamma_1^2 V_k + \gamma_2^2 \|v\|_\infty^2 \\ & < \gamma_1^2 \alpha^k V_0 + \gamma_1^2 \|d\|_\infty^2 + \gamma_2^2 \|v\|_\infty^2, \end{aligned} \quad (36)$$

which implies (23).

### 3.2 Zonotopic analysis

After designing the bank of observers (7), a zonotopic analysis is used to evaluate the FDI consistency test. Assume that the uncertainties are modelled by zonotopes, this method consists in computing a zonotopic residual set relying predominantly on set propagation. Starting from the set of initial residuals, this technique iteratively propagates a sequence of residual sets according to the system dynamics.

If all the residuals generated by the observers belong to the corresponding zonotopes, the system is fault-free. However, once a residual is excluded from the corresponding zonotope, it follows that a fault occurs. If the  $i$ th residual generated by the  $i$ th observer is belong to the corresponding zonotope  $\mathcal{R}_k^i$  and other generated residuals exceed the corresponding zonotopes, it follows that the  $i^*$ th actuator is faulty. The procedure of FDI logic can be summarized in the following:

$$\begin{cases} r_k^i \in \mathcal{R}_k^i, & i = \overline{1, \dots, n_f} \quad (\text{Fault-free}) \\ r_k^i \in \mathcal{R}_k^i \ \& \ r_k^j \notin \mathcal{R}_k^j, & i \neq j \quad (f_i \text{ occurs}) \end{cases} \quad (37)$$

Checking whether the residual  $r_k^i$  is excluded from the the residual zonotope  $\mathcal{R}_k^i$  or not requires solving the following constraint problem:

$$\begin{cases} H_{k,1}^{r,i} z_1^i = 0, \\ \vdots \\ H_{k,n_y}^{r,i} z_{n_y}^i = 0, \\ -1 \leq z_j^i \leq 1, \forall j = \overline{1, \dots, q} \end{cases} \quad (38)$$

where  $H_{k,j}^{r,i}, \forall j = \overline{1, \dots, n_y}$  denotes the  $j$ th component of residual matrix  $H_k^{r,i}$  and  $z_j^i$  represents the  $j$ th component of uncertain vector  $z^i$  bounded by hypercube  $\mathcal{B}^q$  (we consider a zonotope  $\mathcal{R}_k^i$  reduced to dimension  $q$ ).

By means of the interval-hull in Property 1, the zonotope  $\mathcal{R}_k^i$  can be projected into interval bounds as:

$$\begin{cases} \underline{r}_{k,j}^i = -rs(H_k^{r,i})_{jj}, & \forall j = \overline{1, \dots, n_y}, \forall k \in \mathbb{N}, \\ \bar{r}_{k,j}^i = rs(H_k^{r,i})_{jj}, & \forall j = \overline{1, \dots, n_y}, \forall k \in \mathbb{N}, \end{cases} \quad (39)$$

where  $\underline{r}_{k,j}^i$  and  $\bar{r}_{k,j}^i$  are respectively the lower and upper bound of the  $j$ th component of the residual signal  $r^i$ .

Hence the condition  $r_k^i \in \mathcal{R}_k^i$  is equivalent to  $r_k^i \in [\underline{r}_{k,j}^i, \bar{r}_{k,j}^i]$  for  $j = \overline{1, \dots, n_y}$ .

**Theorem 3.** The residual signal  $r_k^i$ , defined in (10), is bounded by a zonotope  $\mathcal{R}_k^i = \langle 0, H_k^{r,i} \rangle$ , in fault-free case ( $f_k = 0$ ), where

$$H_k^{r,i} = [\tilde{C} H_k^{e,i} \ D_2 H_v], \quad (40)$$

and  $H_k^{e,i}$  is given by:

$$\begin{cases} H_{k+1}^{e,i} = [(\tilde{A} - L_i\tilde{C}) \downarrow_q (H_k^{e,i}) \tilde{D}_1 H_w - L_i D_2 H_v]; \\ H_k^{e,i} = \begin{bmatrix} H_k^i & 0 \\ 0 & H_{k-h}^i \end{bmatrix}, \forall k \in [-h, 0]. \end{cases} \quad (41)$$

**Proof.** Based on Assumption 2., we have  $x_k \in \langle p_k^i, \tilde{H}_k^i \rangle$  for all  $k \in [-h, 0]$  with  $\tilde{H}_k^i = \begin{bmatrix} H_k^i & 0 \\ 0 & H_{k-h}^i \end{bmatrix}$ . We assume that  $\hat{x}_k^i = p_k^i, \forall k \in [-h, 0]$ , then the sub-estimation error  $e_{d,k}^i$  is bounded by a zonotope  $e_{d,k}^i \in \langle 0, \tilde{H}_k^i \rangle, \forall k \in [-h, 0]$ . According to the error dynamics in (11), it follows that:

$$\begin{aligned} e_{d,k+1}^i &\in Z_{k+1}^{e,i} = \langle 0, H_{k+1}^{e,i} \rangle, \\ &= (\tilde{A} - L_i\tilde{C}) \odot \langle 0, H_k^{e,i} \rangle \oplus \tilde{D}_1 \odot \langle 0, H_w \rangle \\ &\oplus (-L_i D_2) \odot \langle 0, H_v \rangle. \end{aligned} \quad (42)$$

Using Property 1., the generator matrix  $H_{k+1}^{e,i}$  can be obtained by (41).

Then, from (11),  $r_k^i \in \mathcal{R}_k^i = \langle 0, H_k^{r,i} \rangle$  such that:

$$H_k^{r,i} = [\tilde{C} H_k^{e,i} \ D_2 H_v].$$

#### 4. CASE STUDY

Consider a simple mathematical model of the Mach number dynamic response to guide vane changes described in the work of (Manitius, 1984). The delay in the state variable denotes the transportation time between the guide vanes of the fan and the test section of the tunnel. The discrete version of the nominal model can be reformulated in form of system (5) with the following parameter matrices:

$$\begin{aligned} A &= \begin{bmatrix} 0.9949 & 0 & 0 \\ 0 & 0.9983 & 0.0095 \\ 0 & -0.3431 & 0.9068 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & -0.0016 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B = F &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0.1 & 0.2 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \\ 0.0095 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}. \end{aligned}$$

Then, we can obtain the parameter matrices of the delay-free augmented system with

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} 0.9949 & 0 & 0 & 0 & -0.0016 & 0 \\ 0 & 0.9983 & 0.0095 & 0 & 0 & 0 \\ 0 & -0.3431 & 0.9068 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{D}_1 &= \begin{bmatrix} 0 \\ 0 \\ 0.0095 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B} = \tilde{F} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0.1 & 0.2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \tilde{C} &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Note that, for the first actuator, we have  $\tilde{C}\tilde{F}_1 \neq 0$ . Hence, the associated fault detectability index is  $\rho_1 = 1$ . Therefore, the

fault detectability matrix for the first actuator is obtained by  $\Psi_1 = \tilde{F}_1$ .

Similarly, we have  $\rho_2 = \rho_3 = 1$  Therefore, the fault detectability matrices are obtained by  $\Psi_2 = \tilde{F}_2$  and  $\Psi_3 = \tilde{F}_3$ .

It is clear that  $rank(\tilde{C}\Psi_1) = rank(\tilde{C}\Psi_2) = rank(\tilde{C}\Psi_3) = 1$ , then by letting  $\alpha = 0.3$  and  $\gamma_1 = \gamma_2 = 0.1$  and solving the LMIs in Theorem 2., we obtain:

$$\begin{aligned} L_1 &= \begin{bmatrix} -2.4246 & 2.2042 \\ 3.3519 & -2.1388 \\ -4.2167 & 3.6039 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -2.7564 & 2.2970 \\ 3.6854 & -2.2377 \\ -4.6409 & 3.7327 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ L_3 &= \begin{bmatrix} -4.8835 & 2.4417 \\ 5.8064 & -2.3993 \\ -7.3031 & 3.9334 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

In the simulation, the initial state, is bounded by zonotope as  $x_k \in \langle 0, H_k \rangle$  where

$$H_k : \begin{cases} 0, & k=-h, \dots, -1 \\ 0.5 \times I, & k=0 \end{cases}$$

The disturbance and measurement noise are considered random signals bounded by  $w_k \in \langle 0, 0.5 \rangle$  and  $v(k) \in \langle 0, 0.1 \rangle$ . The control input is chosen as  $u_k = [0.1 \ 0.2 \ 0.3]^T$  and the time-delay is time-invariant with  $h = 6$ . The maximal zonotope order is chosen as 10.

Note that the linear time-delay system is affected by the following additive faults:

$$\begin{aligned} f_1 &= \begin{cases} 0 & k < 25 \\ 0.4 & 25 \leq k < 75 \end{cases}, \quad f_3 = \begin{cases} 0 & k < 200 \\ 0.4 & k \geq 200 \end{cases}, \\ f_2 &= \begin{cases} 0 & k < 125 \\ 0.45 + 0.1\cos(0.1(k)) & 125 \leq k < 175 \end{cases}. \end{aligned}$$

The first and the third actuators are affected by abrupt faults  $f_1$  and  $f_3$  respectively, meanwhile the second actuator is affected by a time-varying fault  $f_2$ .

The simulation has been carried out for  $N_s = 250$  sampling time steps and the simulation results are shown in Fig. 2-4, which show the evolution of residual signals derived from FDO 1, 2 and 3, where solid lines present the residual signals and red and blue dashed lines correspond to the upper and lower bounds of the residual signals computed by (39), respectively. From Fig.2, it can be seen that the residual signals generated from FDO 1 are decoupled from the first fault  $f_1$ , however, they can detect quickly the second and the third faults ( $f_2$  and  $f_3$ ) after their occurrence. According to the proposed FDI strategy, the first actuator is faulty and the fault  $f_1$  is completely isolated.

Similarly, for the FDO 2 (Fig.3), the residual signals are sensitive to  $f_1$  and  $f_3$ . Then, the fault  $f_2$  is completely isolated. For the FDO 3 (Fig.4), the residual signals are sensitive to  $f_1$  and  $f_2$ . Then, the fault  $f_3$  is completely isolated.

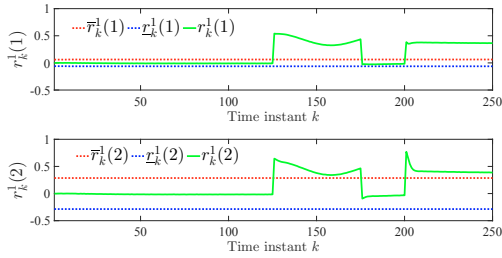


Fig. 2. Residual signals of FDO 1.

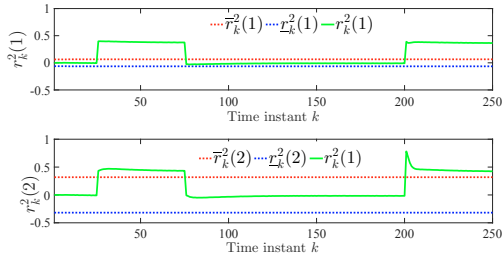


Fig. 3. Residual signals of FDO 2.

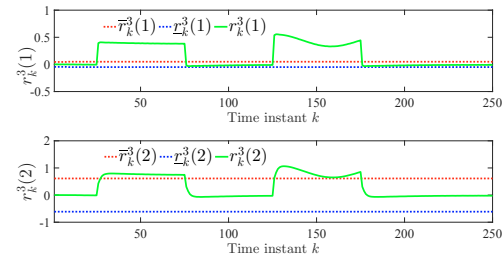


Fig. 4. Residual signals of FDO 3.

## 5. CONCLUSION

This paper proposes a novel FDI scheme based on a bank of  $\mathcal{L}_\infty$  FDOs and zonotopic analysis for linear discrete-time systems with constant time-delay. The disturbances and noises are assumed to be unknown but bounded in given zonotopes. Based on the delay-free augmented system equivalent representation, the FDOs are proposed such that the generated residual signals are robust against disturbances and noises, meanwhile sensitive to faults by investigating an  $\mathcal{L}_\infty$  performance and fault detectability matrix and indices. To achieve fault isolation, the  $i$ th FDO is designed to be decoupled from the  $i$ th actuator fault but sensitive to others. In order to overcome the weakness of predefining threshold for FDI decision-making, a zonotopic analysis method is proposed to evaluate the generated residuals. A case study is considered to show the efficiency of the proposed approach. For future works, extension of these results to the case of time-varying delay and uncertain delays will be considered.

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