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A hybrid lattice Boltzmann - Navier-Stokes method for unsteady aerodynamic and aeroacoustic computations

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Abstract

A hybrid numerical method coupling the standard lattice Boltzmann method (LBM) and a compressible finite-volume Navier-Stokes (NS) solver is proposed in the context of unsteady aerodynamic and aeroacoustic simulations. The trend being towards more realistic and detailed simulations in a reasonable amount of CPU time, lattice Boltzmann and Navier-Stokes solvers can be combined to solve the same problem. The present hybrid method relies on a zonal decomposition of the computational domain thus allowing to exploit the numerical features of both methods to their optimal extent in specific flow regions.

The key issue when combining the LBM with a Navier-Stokes solver is to ensure a smooth transition of the flow variables at the two-way coupling interface. While existing approaches consider overlapping meshes, a direct grid coupling is here derived. The mapping from the macroscopic flow variables to the set of lattice Boltzmann distribution functions is performed analytically thanks to a Chapman-Enskog expansion and draws a direct link to advanced regularised collision operators. Unsteady computations are enabled by coupling the lattice Boltzmann stream and collide algorithm with explicit and implicit Navier-Stokes time-stepping schemes. The hybrid method is then assessed on four time-dependent test cases representative of aerodynamic and aeroacoustic problems. The proposed approach is proven to yield very accurate results while keeping the numerical advantages of both methods and reducing the overall computational cost of direct noise computations.

Keywords: Lattice Boltzmann Method, Navier-Stokes equations, Finite-Volume, Coupling, Aerodynamics, Computational Aeroacoustics

1. Introduction

Computational Fluid Dynamics (CFD) has become an important tool in aerospace sciences enabling both researchers and engineers to get more insight into complex fluid phenomena and drastically decrease

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4 aircraft development lead-time [1]. Nowadays, Reynolds-Averaged Navier-Stokes (RANS) simulations on
5 body-fitted meshes are of common practice in the industry since computations can easily be carried out
6 overnight. However, RANS solutions, which rely on the modelling of all turbulent scales, are unreliable if
7 complex turbulent phenomena occur or if aeroacoustics have to be finely characterised. As a consequence,
8 the ability to perform accurate broadband three-dimensional unsteady flow simulations such as Large Eddy
9 Simulations (LES) in reasonable computational time is a crucial issue.

10 In his 2019 review paper, Löhner [2] suggested that structured finite-type Navier-Stokes methods or
11 lattice Boltzmann methods might be the most promising ones to achieve industrial level LES simulations
12 in the next few years. On one hand, LES or hybrid RANS-LES applied to the Navier-Stokes equations
13 are established approaches to describe the behavior of turbulent flows involving complex geometries [3].
14 Nevertheless, Navier-Stokes finite-type methods intrinsically suffer from numerical dissipation and thus face
15 some difficulties to accurately predict the transport of turbulence over long distances and far-field acoustics.
16 On the other hand, the lattice Boltzmann method (LBM) [4, 5, 6] which relies on a mesoscopic description of
17 collisions between fluid particles, has gained an increasing amount of attention in the last decades. Indeed,
18 the LBM appears as a good candidate for capturing the small acoustic pressure fluctuations thanks to its low
19 dissipative properties [7]. Moreover, the LBM also provides the advantage of having a lower computational
20 cost per mesh point with respect to traditional Navier-Stokes methods [7]. In light of these advantages, its
21 range of applicability in both physics and engineering has grown in such a way that it enabled the simulation
22 of a variety of very complex phenomena in aeronautics [8, 9, 10]. However, the standard lattice Boltzmann
23 models still suffer from some limitations such as their restriction to low compressible isothermal flows and,
24 by construction, their restriction to Cartesian grids and explicit time-stepping at constant CFL number.
25 Consequently, the computation of wall-resolved turbulent boundary layers within the lattice Boltzmann
26 framework remains costly and, despite their lack of generality and modeling errors [11], the use of wall-models
27 is of common practice. Hence, in near-wall flow regions, solving a discretised form of the Navier-Stokes
28 equations might outperform the LBM benefiting from their great flexibility through the use of body-fitted
29 anisotropic meshes or implicit time-stepping. In other words, depending on the nature of the flow region,
30 optimal efficiency may be reached with a different solver.

31 Following this idea, few hybrid lattice Boltzmann - Navier-Stokes methods for fluid problems have been
32 proposed over the years. As indicated in a recent review [12], they are all based on a state-exchange with
33 overlapping regions where both macroscopic and mesoscopic variables are computed. The key issue when
34 coupling both solvers is to understand how the lattice Boltzmann set of variables is related to the Navier-
35 Stokes one and conversely. Going from the LBM variables to the macroscopic ones is naturally offered by
36 the statistical moments of distribution functions. However, the inverse mapping is not univocal as the dis-
37 tribution functions outnumber the macroscopic variables. The first coupling between the lattice Boltzmann

38 method and an incompressible finite-difference Navier-Stokes solver for fluid problems was introduced by
39 Latt *et al.* [13] following the theoretical basis of Albuquerque *et al.* [14]. They proposed to use a Chapman-
40 Enskog expansion leading to analytical passage relations between macroscopic quantities and distribution
41 functions. Despite the fact that some restrictive assumptions were made in the Chapman-Enskog expansion
42 by neglecting high-order terms and temporal derivatives, the method was validated on a $2D$ steady
43 Poiseuille flow. Later, Luan *et al.* [15] introduced a coupled lattice Boltzmann - finite-volume Navier-Stokes
44 method for convective heat transfer and incompressible fluid dynamics problems. The originality of their
45 work relies in a reconstruction procedure specifically designed for the BGK collision operator [16]. However,
46 in their strategy, the lattice Boltzmann method was used in near-wall regions, thus leading to an inadequate
47 resolution of the boundary layer around a NACA0012 airfoil due to modelling errors [17]. Moreover, some
48 discontinuities in the vicinity of the coupling interfaces have been evidenced in both the vorticity and pressure
49 fields at steady state. They were expected to be caused by the weak compressibility and unsteady nature of
50 the LBM compared to the incompressible and steady Navier-Stokes solver used in their study. To overcome
51 the lack of generality in the prior study, Tong *et al.* [18] introduced a generalised reconstruction operator
52 but still neglected high-order derivatives of distribution functions without any rigorous explanation. Their
53 methodology was the first to be applied on unsteady fluid flow problems and relied on lattice Boltzmann
54 sub-iterations in order to damp out potential spurious pressure oscillations. At the same time, Neumann
55 *et al.* [19, 20, 21] also developed a steady and unsteady hybrid lattice Boltzmann – Navier-Stokes method
56 using another macro-to-meso mapping strategy. Indeed, in their approach, the distribution functions are
57 sought as solutions of an optimisation problem under conservation constraints. Despite promising results in
58 the steady case [20], the extension to unsteady test cases was found to lead to compressibility errors large
59 enough to severely perturb both the pressure and velocity field in the whole computational domain [21].

60 The present work follows the path paved by Albuquerque and Latt [13, 14] and further extends it to the
61 simulation of unsteady flows. To ensure continuous pressure and vorticity fields, it is proposed to couple
62 two compressible flow solvers, one relying on a finite-volume discretisation of the Navier-Stokes equations
63 and another one based on the standard lattice Boltzmann method. The proposed coupling procedure does
64 not require overlapping between the solvers and is introduced in a general context without any restrictive
65 assumptions on the distribution functions. Thanks to the use of an advanced regularised collision operator
66 [22], the coupling boundary condition is found to be directly included in the LB scheme thus requiring no
67 additional computations or storage. Furthermore, a coupling between the LBM and other time advance
68 schemes is introduced and thoroughly discussed in order to preserve the flexibility of the time-integration
69 offered by the use of NS solvers.

70 The paper is organised as follows. In Section 2, the numerical methods adopted for the present study,
71 namely the Navier-Stokes finite-volume (NS-FV) solver and the lattice Boltzmann method with HRR col-

72 lision operator, are presented. Then, the coupling procedure between both solvers is described in Section
73 3 which covers: (1) the rescaling of flow quantities, (2) a specific thermodynamic closure, (3) the recon-
74 struction of distribution functions, and (4) the coupling of time-marching schemes. In Section 4, numerical
75 validations are performed on academic test cases. The capabilities of the hybrid lattice Boltzmann - Navier-
76 Stokes method are finally fully demonstrated on the aeroacoustic study of the flow past a circular cylinder
77 on a hybrid curvilinear-Cartesian grid. This last test case will highlight the ability of the present hybrid
78 method to perform simultaneously wall- and acoustically-resolved simulations in a competitive CPU time.

79 2. Numerical methods

80 In this section, the numerical methods used in the framework of the lattice Boltzmann - Navier-Stokes
81 coupling are briefly introduced. All the methods presented hereafter are part of the FAST (Flexible Aero-
82 dynamic Software Technology) CFD suite developed at ONERA [23] which consists of Python modules
83 implementing High Performance Computing (HPC) dedicated solvers for unsteady fluid dynamics applica-
84 tions. In addition, all the pre- and post-processing tasks were performed using Cassiopee [24].

85 2.1. The finite-volume Navier-Stokes flow solver

86 The three-dimensionnal compressible unsteady Navier-Stokes (NS) equations are solved using ONERA's
87 FastS solver dedicated to multi-block structured grids. Starting from the conservative form of the
88 Navier-Stokes equations:

$$\frac{\partial}{\partial t} \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) - \nabla \cdot \mathbf{F}^\nu(\mathbf{U}) = \mathbf{0}, \quad (1)$$

89 where $\mathbf{U} = (\rho, \rho u_i, \rho E)^t$, $\mathbf{F}(\mathbf{U})$ and $\mathbf{F}^\nu(\mathbf{U})$ are the flow variable vectors, the inviscid and viscous fluxes,
90 respectively; the cell-centered finite volume method is obtained by splitting the computational domain Ω
91 into N non-overlapping cells Ω_{ijk} . The integration of equation (1) over every cell of the mesh leads to a
92 semi-discrete form as:

$$\frac{d}{dt} \mathbf{U}_{ijk} + \frac{1}{|\Omega_{ijk}|} \mathbf{R}_{ijk}(\mathbf{U}) = \mathbf{0}, \quad (2)$$

93 where \mathbf{U}_{ijk} is now the mean flow variable vector evaluated at the center of Ω_{ijk} , $|\Omega_{ijk}|$ the volume of Ω_{ijk}
94 and \mathbf{R}_{ijk} the residual of the discretised convective and viscous terms. Actually, the residual is defined as
95 the algebraic sum of the convective and viscous fluxes over the whole boundary of a cell.

96 The convective fluxes are being approximated with a second-order accurate scheme proposed by Mary &
97 Sagaut [25]. It relies on a hybrid centered/decentered modification of the AUSM+(P) scheme (see Edwards
98 & Liou [26]) offering a good trade-off between robustness, accuracy, and computational cost. The viscous
99 fluxes are discretised by a second-order accurate centered scheme. The time integration can be carried out
100 by means of an explicit 3^{rd} -order accurate low-storage Runge-Kutta scheme [27] or by means of the implicit
101 2^{nd} -order accurate backward scheme of Gear with local Newton sub-iterations [28]. In the case of the implicit

102 time-stepping scheme, the Jacobians are approximated following the procedure presented in [29, 30] and the
 103 linear system is solved by the LU-SGS factorisation [29].

104 FastS solver has been extensively used and validated for both academic and industrial unsteady flow
 105 simulations such as transitional separation bubble [31], airfoils in near stall configurations [25, 32] and
 106 laminar transonic buffet [33]. One major feature of this solver is its computational efficiency since it enables
 107 to update over one complete time step up to 3.5 million cells per second and per core on a single Intel
 108 Broadwell processor [34].

109 2.2. The lattice Boltzmann method

110 2.2.1. Basics of the lattice Boltzmann method

111 Unlike the finite-volume method described in Section 2.1, the lattice Boltzmann method [4, 5, 6] does
 112 not directly rely on the resolution of the Navier-Stokes equations. In fact, it originates from a very specific
 113 discretisation of the Boltzmann equation describing the evolution of gases in terms of distribution functions
 114 $f(\mathbf{x}, \boldsymbol{\xi}, t)$. These can be viewed as representing the probability density of finding fictive particles at a location
 115 \mathbf{x} and time t being advected at a given velocity $\boldsymbol{\xi}$. In the absence of a body-force term, the Boltzmann
 116 equation is given by:

$$\frac{\partial}{\partial t} f(\mathbf{x}, \boldsymbol{\xi}, t) + \boldsymbol{\xi} \cdot \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \boldsymbol{\xi}, t) = -\frac{1}{\tau} (f - f^{eq}), \quad (3)$$

117 where the BGK collision operator [35] has been used to model the time evolution of the distribution functions
 118 due to collisions between particles. The main idea behind the BGK model is to describe the collisions only
 119 through their average effect which can be viewed as a relaxation process towards a local equilibrium f^{eq}
 120 with a single relaxation time τ .

121 In order to solve (3), one should not only discretise space and time like in standard NS solvers, but also
 122 the velocity space. This step is of utmost importance and gives the LBM its originality. The main idea
 123 is to restrict the continuous velocity space to a finite set of q velocities $\{\boldsymbol{\xi}_i\}_{i \in \llbracket 1; q \rrbracket}$ so that the macroscopic
 124 behavior of the Navier-Stokes equations is still recovered. To perform this discretisation, a standard Gauss-
 125 Hermite quadrature is commonly employed [36, 37]. As a result, the Boltzmann equation is now solved for
 126 a discrete set of distribution functions $\{f_i\}_{i \in \llbracket 1; q \rrbracket}$ where f_i is associated with the discrete velocity $\boldsymbol{\xi}_i$. The
 127 resulting equation is called the discrete velocity Boltzmann equation (DVBE) and reads:

$$\frac{\partial}{\partial t} f_i(\mathbf{x}, t) + \boldsymbol{\xi}_i \cdot \frac{\partial}{\partial \mathbf{x}} f_i(\mathbf{x}, t) = -\frac{1}{\tau} (f_i - f_i^{eq}) \quad i \in \llbracket 1; q \rrbracket. \quad (4)$$

128 The set of discrete velocities $\{\boldsymbol{\xi}_i\}_{i \in \llbracket 1; q \rrbracket}$ is often referred to as a $DdQq$ lattice, where d is the spatial
 129 dimension and q the number of discrete velocities. For this study, the usual D3Q19 lattice represented on
 130 Figure 1 is used. It is defined by a set of 19 discrete velocities $\boldsymbol{\xi}_i$, their associated Gaussian weights w_i , and
 131 by a lattice constant c_s corresponding to the lattice speed of sound.

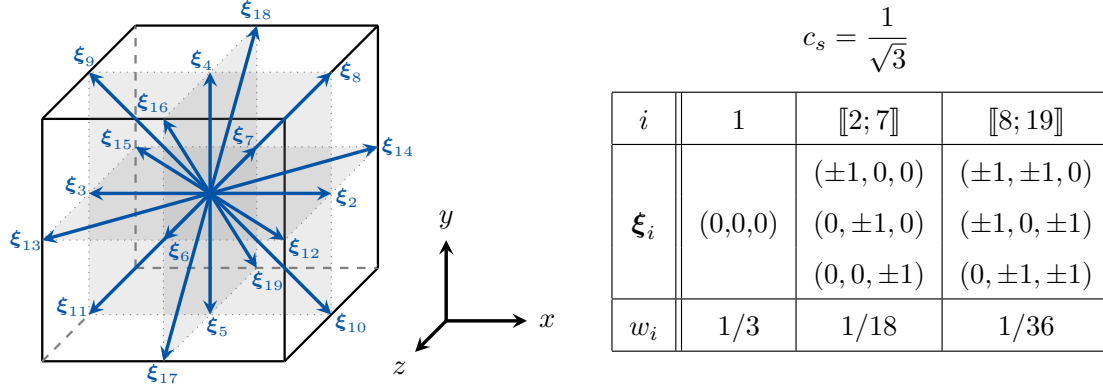


Figure 1: D3Q19 velocity set. The cube, drawn in solid lines, has an edge length of $2\Delta x$. For the sake of clarity, the rest velocity $\xi_1 = \mathbf{0}$ is not represented as it lies at the center of the cube. Each discrete velocity ξ_i is expressed in its non-dimensional form.

132 The macroscopic quantities of interest for an athermal flow such as the density ρ and the velocity field
 133 \mathbf{u} can be deduced from the set of discrete distribution functions by taking their moments:

$$\rho(\mathbf{x}, t) = \sum_{i=1}^q f_i(\mathbf{x}, t) = \sum_{i=1}^q f_i^{eq}(\mathbf{x}, t) \quad \rho \mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^q \xi_i f_i(\mathbf{x}, t) = \sum_{i=1}^q \xi_i f_i^{eq}(\mathbf{x}, t). \quad (5)$$

134 Nevertheless, restricting the velocity space to only 19 discrete velocities has an impact on the macroscopic
 135 equations recovered by the DVBE and consequently by the LBM. As shown in [37] this gives rise to an
 136 athermal flow hypothesis (i.e. $T = T_0$ leading to a speed of sound $c_0^{ath} = \sqrt{RT_0}$ with R the specific
 137 gas constant) as well as a cubic Mach error term in the momentum equations [4, 38] thereby limiting the
 138 application of standard lattice Boltzmann methods to weakly compressible and low-mach number flows.

139 In order to obtain the well-known LBM “Stream & Collide” algorithm, the space and time discretisation
 140 of the DVBE (4) has to be performed. Thanks to its mathematical structure, the left-hand side (LHS) linear
 141 convection term of Equation (4) is integrated along the ξ_i characteristic ensuring an exact advection step
 142 and a direct link between the grid and time step through $\Delta x = |\xi_i| \Delta t$. On the other hand, a trapezoidal
 143 integration rule is employed for the right-hand side (RHS) collision term. This strategy, leads to a system
 144 of two equations :

$$\begin{cases} g_i(\mathbf{x} + \xi_i \Delta t, t + \Delta t) = g_i^{coll}(\mathbf{x}, t), \\ g_i^{coll}(\mathbf{x}, t) = g_i(\mathbf{x}, t) - \frac{\Delta t}{\bar{\tau}} (g_i(\mathbf{x}, t) - g_i^{eq}(\mathbf{x}, t)), \end{cases} \quad (6)$$

145 where $\bar{\tau} = \tau + \frac{\Delta t}{2}$ and $\{g_i\}_{i \in [1; q]}$ are the modified distribution functions so as to ensure an explicit formulation
 146 of the algorithm [39]. They are related to the original distribution functions $f_i(\mathbf{x}, t)$ through the relation

$$g_i(\mathbf{x}, t) = f_i(\mathbf{x}, t) + \frac{\Delta t}{2\tau} (f_i(\mathbf{x}, t) - f_i^{eq}(\mathbf{x}, t)), \quad (7)$$

147 which implies $g_i^{eq}(\mathbf{x}, t) = f_i^{eq}(\mathbf{x}, t)$. In Equation (6), the *coll* superscript refers to the post-collision state.

148 The lattice Boltzmann method is classically applied in a non-dimensional form. Therefore, the time-
 149 step Δt (resp. the grid-step Δx) is chosen as the characteristic time (resp. characteristic length) for the
 150 non-dimensionalization. This leads to the lattice Boltzmann scheme:

$$g_i(\mathbf{x} + \boldsymbol{\xi}_i, t + 1) = g_i(\mathbf{x}, t) - \frac{\Delta t}{\bar{\tau}} (g_i(\mathbf{x}, t) - g_i^{eq}(\mathbf{x}, t)). \quad (8)$$

151 An important point when focusing on acoustics is the physical speed of sound simulated by the LBM denoted
 152 hereafter by c_0^{LBM} . Indeed, the athermal sound speed $c_0^{\text{ath}} = \sqrt{RT_0}$ imposed by the D3Q19 lattice does not
 153 correspond to the expected one $c_0 = \sqrt{\gamma RT_0}$ where γ is the heat capacity ratio. However, in practice, one
 154 can enforce $c_0^{\text{LBM}} = c_0$ by computing the time-step following the acoustic scaling. The physical values of the
 155 speed of sound c_0^{LBM} and the viscosity ν are then given by:

$$c_0^{\text{LBM}} = c_0 = \frac{c_s \Delta x}{\Delta t} \quad \text{and} \quad \nu = (c_0^{\text{LBM}})^2 \left(\bar{\tau} - \frac{1}{2} \right), \quad (9)$$

156 where $c_s = 1/\sqrt{3}$ is the D3Q19 lattice constant. Note that the acoustic scaling is equivalent to setting the
 157 LBM fictitious temperature to γT_0 where γ is the heat capacity ratio of the fluid to be simulated.

158 Based on these parameters, the LBM recovers the athermal and low-compressible Navier-Stokes dynamics
 159 with a second-order accuracy in both space and time [4, 40].

160 2.2.2. The Hybrid Recursive Regularised collision operator

161 The basic lattice Boltzmann method with the single relaxation time BGK collision model presented
 162 in Section 2.2.1 suffers from stability issues especially in the low viscosity regime (i.e. at high Reynolds
 163 number) [41]. These issues have been attributed to interactions between so called “non-hydrodynamic”
 164 modes arising from the space and time discretisation of Equation (4) [42, 43, 44]. To alleviate this problem,
 165 a great number of collision models have been proposed such as Multiple Relaxation Times (MRT) operators
 166 [41, 45, 46], entropic LBMs [47], and regularised approaches [48, 49]. Some authors also suggest to employ
 167 selective filters [50, 51] in order to remove the high wave number instabilities without affecting the large
 168 scale dynamics. The present work focuses on the former regularisation strategy as it can easily be linked to
 169 the coupling methodology as will be seen in Section 3.4.

170 The Hybrid Recursive Regularised (HRR) collision operator [22] will be employed hereafter. Following
 171 the idea of Latt and Chopard [48], before the collision step, distribution functions are regularised as:

$$g_i^{\text{reg}} = g_i^{\text{eq}} + g_i^{(1),\text{reg}}, \quad (10)$$

172 where $g_i^{\text{eq}} = f_i^{\text{eq}}$ is the equilibrium distribution function and $g_i^{(1),\text{reg}}$ the regularised contribution based on
 173 the 1st-order off-equilibrium term of the Chapman-Enskog analysis. The lattice Boltzmann scheme with
 174 HRR collision operator reads:

$$g_i(\mathbf{x} + \boldsymbol{\xi}_i, t + 1) = g_i^{\text{eq}}(\mathbf{x}, t) + \left(1 - \frac{1}{\bar{\tau}} \right) g_i^{(1),\text{reg}}(\mathbf{x}, t) + \frac{1}{2} \psi_i(\mathbf{x}, t). \quad (11)$$

175 In Equation (11), a corrective term denoted by ψ_i is introduced following [52]. Although this term was
 176 initially intended to correct the cubic error of the LBM with a D3Q19 lattice, it was shown recently that
 177 the ψ_i corrective term is mandatory to ensure the stability of the HRR model in the low-viscosity regime
 178 [53]. This correction reads:

$$\psi_i = -w_i \frac{\mathcal{H}_{i,\alpha\beta}^{(2)}}{2c_s^4} \frac{\partial \Psi_{\alpha\beta\gamma}}{\partial x_\gamma} \quad (12)$$

179 where $\mathcal{H}_{i,\alpha\beta}^{(2)} = \xi_\alpha \xi_\beta - c_s^2 \delta_{\alpha\beta}$ is the second-order discrete Hermite polynomial, and $\Psi_{\alpha\beta\gamma}$ is the deviation term
 180 between of the third-order moment of the velocity-discrete equilibrium and its continuous counterpart. In
 181 the present study, the derivatives are estimated with a second-order centered finite-difference scheme. The
 182 reader is referred to [52] for an in-depth discussion of this corrective term.

183 In the same way as for the recursive regularised collision model of Malaspina [49], both the equilibrium
 184 and the regularised off-equilibrium distribution functions are expanded using the Hermite formalism:

$$g_i^{eq} = w_i \sum_{n=0}^N \frac{1}{c_s^{2n} n!} \mathbf{a}_0^{(n)} : \mathcal{H}_i^{(n)} \quad \text{and} \quad g_i^{(1),reg} = w_i \sum_{n=2}^{N_r} \frac{1}{c_s^{2n} n!} \mathbf{a}_1^{(n)} : \mathcal{H}_i^{(n)}, \quad (13)$$

185 where “:” stands for the full contraction of indices of two n^{th} -order tensors: the Hermite coefficients $\mathbf{a}_0^{(n)}$
 186 and $\mathbf{a}_1^{(n)}$, and the discrete Hermite polynomials $\mathcal{H}_i^{(n)} = \mathcal{H}^{(n)}(\boldsymbol{\xi}_i)$. In the present work, $N = N_r = 3$. It is
 187 also worth noting that, as introduced in [22], judicious combinations of third-order Hermite polynomials are
 188 used to build the third-order contributions in g_i^{eq} and $g_i^{(1),reg}$, leading to:

$$\begin{aligned} \mathbf{a}_\alpha^{(3)} : \mathcal{H}_i^{(3)} = & 3 \left(\mathcal{H}_{i,xxxy}^{(3)} + \mathcal{H}_{i,yzz}^{(3)} \right) \left(a_{\alpha,xxxy}^{(3)} + a_{\alpha,yzz}^{(3)} \right) + \left(\mathcal{H}_{i,xxxy}^{(3)} - \mathcal{H}_{i,yzz}^{(3)} \right) \left(a_{\alpha,xxxy}^{(3)} - a_{\alpha,yzz}^{(3)} \right) \\ & + 3 \left(\mathcal{H}_{i,xzz}^{(3)} + \mathcal{H}_{i,xyyy}^{(3)} \right) \left(a_{\alpha,xzz}^{(3)} + a_{\alpha,xyyy}^{(3)} \right) + \left(\mathcal{H}_{i,xzz}^{(3)} - \mathcal{H}_{i,xyyy}^{(3)} \right) \left(a_{\alpha,xzz}^{(3)} - a_{\alpha,xyyy}^{(3)} \right) \\ & + 3 \left(\mathcal{H}_{i,yyz}^{(3)} + \mathcal{H}_{i,xxz}^{(3)} \right) \left(a_{\alpha,yyz}^{(3)} + a_{\alpha,xxz}^{(3)} \right) + \left(\mathcal{H}_{i,yyz}^{(3)} - \mathcal{H}_{i,xxz}^{(3)} \right) \left(a_{\alpha,yyz}^{(3)} - a_{\alpha,xxz}^{(3)} \right), \end{aligned} \quad (14)$$

189 for $\alpha = 0$ and 1. By definition, the Hermite polynomials $\mathcal{H}_{i,\gamma\gamma\delta}^{(3)}$ are given by $\mathcal{H}_{i,\gamma\gamma\delta}^{(3)} = \xi_{i\gamma}^2 \xi_{i\delta} - c_s^2 \xi_{i\delta}$. The
 190 formulas of the equilibrium and off-equilibrium Hermite coefficients are provided in Table 1.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$\mathbf{a}_0^{(n)}$	$a_0^{(0)} = \rho$	$a_{0,\alpha}^{(1)} = \rho u_\alpha$	$a_{0,\alpha\beta}^{(2)} = \rho u_\alpha u_\beta$	$a_{0,\alpha\beta\gamma}^{(3)} = \rho u_\alpha u_\beta u_\gamma$
$\mathbf{a}_1^{(n)}$	$a_1^{(0)} = 0$	$\mathbf{a}_1^{(1)} = \mathbf{0}$	Equation (15)	$a_{1,\alpha\beta\gamma}^{(3)} = u_\alpha a_{1,\beta\gamma}^{(2)} + u_\beta a_{1,\alpha\gamma}^{(2)} + u_\gamma a_{1,\alpha\beta}^{(2)}$

Table 1: Formulas of the equilibrium and off-equilibrium Hermite coefficients. All these expressions, except the one for $\mathbf{a}_1^{(2)}$, were derived using Malaspina’s recursive formulas [49].

191 The very essence of the HRR collision operator lies in the way the second-order off-equilibrium Hermite
 192 coefficient $\mathbf{a}_1^{(2)}$ is computed [22]. In the HRR framework, this tensor is hybridised. It is decomposed into a

193 linear combination of a projected regularised part and a finite difference part, yielding:

$$\mathbf{a}_1^{(2)} = \sigma \left[\sum_{i=1}^q \mathcal{H}_i^{(2)} \left(g_i - g_i^{eq} + \frac{\psi_i}{2} \right) \right] + (1 - \sigma) \left[-\rho \bar{c}_s^2 \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t \right) \right] \quad \text{with } 0 \leq \sigma \leq 1. \quad (15)$$

194 The spatial derivatives of the velocity field present in Equation (15) are evaluated with second-order centered
 195 finite differences. σ is a user-tuned parameter to control the amount of hyper-viscosity added to the model
 196 [22]. In the remainder of this paper the value $\sigma = 0.995$ is adopted since numerical tests have indicated
 197 that this specific value of σ allows to obtain stable computations while limiting the numerical dissipation of
 198 shear and acoustic waves to a very small extent.

199 3. Lattice Boltzmann - Navier-Stokes coupling

200 Having presented the main components of the lattice Boltzmann and finite-volume solvers used for this
 201 study, the focus is now made on the coupling between both methods. First, some general notions about the
 202 coupling interface are introduced. This will help to highlight the main issues to resolve before setting up a
 203 coupled simulation. As a result of this discussion, the rescaling step, the thermodynamic closure as well as
 204 the reconstruction of the distribution functions and the temporal coupling are introduced and discussed.

205 3.1. The coupling interface

206 To illustrate the basic idea of the lattice Boltzmann - Navier-Stokes coupling, a simplified one-dimensional
 207 case represented on Figure 2 is studied. The computational domain denoted by Ω is decomposed into two
 208 sub-domains Ω_{NS} and Ω_{LBM} such that $\Omega = \Omega_{\text{NS}} \cup \Omega_{\text{LBM}}$ and $\Omega_{\text{NS}} \cap \Omega_{\text{LBM}} = \emptyset$. The finite volume method
 209 is applied in Ω_{NS} and the lattice Boltzmann method is applied in Ω_{LBM} . In the following, both solvers are
 210 supposed to be cell-centered as it the case for ONERA’s FAST CFD suite. Moreover, it is also assumed that
 211 both domains have the same grid size Δx at least in the vicinity of the interface. The following methodology
 212 can then be extended with few changes to any more general boundary. Indeed, when the cells centers do
 213 not coincide at the interface, spatial interpolation has to be added to the following procedure.

214 The finite volume scheme is applied on the Navier-Stokes Finite-Volume (NS-FV) interior cells (○). To
 215 be consistent with the finite volume solver presented in Section 2.1 which is based on a five-point stencil,
 216 two layers of ghost cells are required at the borders of the Ω_{NS} sub-domain to specify the coupling boundary
 217 condition. The information transfer from the LBM domain to the NS domain (→) then consists in
 218 imposing the flow variable vector $\mathbf{W} = (\rho, u_i, T)^t$ at the NS-FV boundary nodes (○). With the help of
 219 Equation (5), the first macroscopic variables of \mathbf{W} are easily obtained by taking the discrete moments of the
 220 distribution functions in the corresponding LBM donor cells (□). However, some attention has to be paid to
 221 the fact that the LBM solver has its own system of “lattice units” in contrast to the NS-FV solver. Moreover,
 222 as evidenced in Section 2.2.1, the use of a D3Q19 lattice imposes a constant temperature $T = T_0$ and a

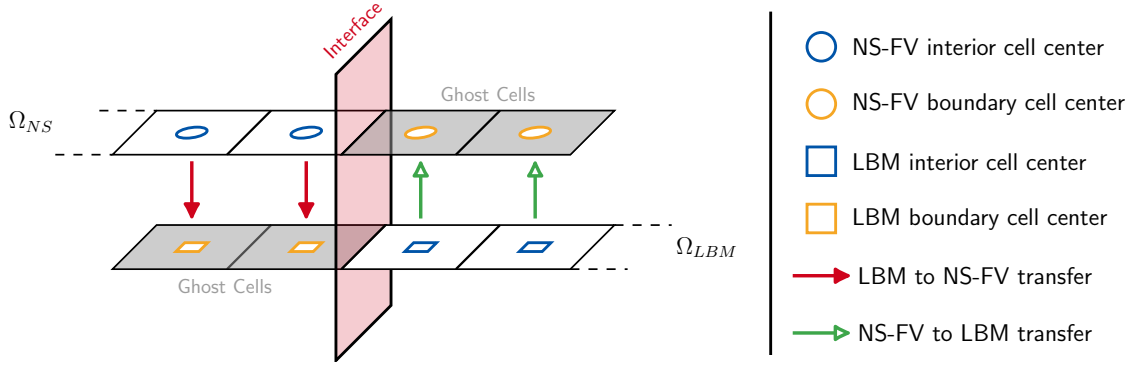


Figure 2: One dimensional representation of a coupling interface. The computational domain is decomposed into a finite-volume Navier-Stokes sub-domain Ω_{NS} and a LB sub-domain Ω_{LBM} . No overlapping region where both methods are simultaneously applied is needed. The information exchange is only ensured through the ghost-cells.

223 barotropic equation of state which does not coincide with the perfect gas law simulated by the finite-volume
 224 method. As a consequence, a rescaling step and a temperature closure are needed to fully determine the
 225 LBM to NS-FV transfer (\rightarrow). This will be the topic of Sections 3.2 and 3.3.

226 Symmetrically, the lattice Boltzmann “Stream & Collide” algorithm is applied on the LBM interior cells
 227 (\square) while the ghost cells (\square) are used to specify the coupling boundary condition. Here again, two layers of
 228 ghost cells are defined. The information transfer from the NS domain to the LBM domain (\rightarrow) is not as
 229 straightforward as the inverse owing to the fact that the information provided by the donor cells (\circ) of the
 230 FV solver (i.e. the 5 components of the flow variable vector $\mathbf{W} = (\rho, u_i, T)^t$) only represents a subset of the
 231 information needed on the scale of the distribution functions (i.e. the 19 distribution functions for a D3Q19
 232 lattice). Hence, a one-to-many problem arises. Therefore, it is of utmost importance to understand how to
 233 link the macroscopic variables to the LB set of distribution functions. This specific issue is at the heart of
 234 the coupling procedure and will be addressed in Section 3.4.

235 To end this discussion on the simplified case of Figure 2, a remark has to be made regarding the time
 236 advance procedure in the presence of a coupling interface. By construction, the LBM is an explicit scheme
 237 with constant CFL number, unlike the finite volume method which offers various time-stepping options. As
 238 a consequence, it might be very likely that the two schemes do not use the same time stepping strategy. In
 239 Section 3.5, a methodology to bridge the LBM with other time advance schemes is proposed.

240 3.2. Rescaling of macroscopic flow quantities

241 At the coupling interface between the lattice Boltzmann and the Navier-Stokes solver, only macroscopic
 242 quantities (i.e. the 5 components of the flow state vector $\mathbf{W} = (\rho, u_i, T)^t$) will be exchanged between
 243 the solvers inasmuch as they are the only variables in common to both numerical methods. If the cells
 244 centers coincide at the interface, it is sufficient to copy the values from one grid to the other, but if not, an

245 interpolation step is required in order to evaluate the variables at the centers of the receiving grid. Now,
 246 regardless of the transfer type, a rescaling step has to be performed for each exchange since the lattice
 247 Boltzmann and Navier-Stokes solvers are implemented in different systems of units. Hereafter, any variable
 248 v expressed in lattice units will be denoted by \tilde{v} while \bar{v} refers to its Navier-Stokes non-dimensional value.

249 As stated in Section 2.2, the lattice Boltzmann method is expressed in a very specific set of units often
 250 referred to as the “lattice units” where the grid- and time-step Δx and Δt are used as characteristic length
 251 and time scales for the non-dimensionalization. The density field is also made non-dimensional so that, in
 252 average, $\tilde{\rho}^{\text{LBM}} = \rho^{\text{LBM}}/\rho_0 = 1$ where ρ_0 is the density scaling factor. In the same way, it is of common
 253 practice to use dimensionless quantities for a finite-volume Navier-Stokes solver. However it should be noted
 254 that the conversion factors in the NS framework are most of the time chosen as flow dependent quantities
 255 (the free-stream velocity U_∞ , density ρ_∞ , and a geometrical characteristic length L_0 for instance) rather
 256 than numerical parameters such as Δx and Δt . Consequently when exchanging data between the LB and
 257 NS-FV solvers, the following scaling formulas have to be applied:

$$\bar{\rho}^{\text{NS}} = \frac{\rho_0 \tilde{\rho}^{\text{LBM}}}{\rho_\infty} = \frac{\rho_0}{\rho_\infty} \sum_{i=1}^q g_i(\mathbf{x}, t) \quad \text{and} \quad \bar{u}_i^{\text{NS}} = \frac{\tilde{u}_i^{\text{LBM}} \frac{\Delta x}{\Delta t}}{U_\infty} = \frac{\Delta x}{U_\infty \Delta t \tilde{\rho}^{\text{LBM}}} \sum_{j=1}^q \xi_j g_j(\mathbf{x}, t). \quad (16)$$

258 In Equation (16), Δt and Δx refer to the time- and grid-steps of the donor solver. Obviously, the case of a
 259 dimensional NS-FV solver is directly obtained by setting $U_\infty = \rho_\infty = 1$.

260 The last variable which has to be provided to the NS-FV solver, is the temperature. However, as already
 261 stated, the D3Q19 lattice employed by the LB solver only solves an athermal version of the Navier-Stokes
 262 equations and thus imposes a constant temperature $T = T_0$. In order to alleviate this issue, some work on
 263 the thermodynamic closure has to be done.

264 3.3. Thermodynamic closure

265 All the previous work done on coupled LB-NS simulations relied on an incompressible Navier-Stokes
 266 flow solver [13, 15, 20, 21]. Consequently, only the velocity field had to be provided to the NS solver while
 267 the pressure field was computed via the embedded Poisson solver. Conversely, the density field which is
 268 needed for the LBM was directly computed from the NS pressure field. Extending this methodology to a
 269 compressible Navier-Stokes solver is not as straightforward.

270 The present methodology suggests to use pressure as an intermediate variable to compute temperature
 271 fluctuations around its reference value T_0 . The virtual fluid simulated by the standard lattice Boltzmann
 272 method relies on a barotropic equation of state which reads as:

$$p^{\text{LBM}} = \rho (c_0^{\text{LBM}})^2 = \rho_0 (c_0^{\text{LBM}})^2 + \rho' (c_0^{\text{LBM}})^2 = p_0^{\text{LBM}} + \rho' (c_0^{\text{LBM}})^2, \quad (17)$$

273 where the density has been decomposed as $\rho = \rho_0 + \rho'$ (ρ_0 being the reference density and ρ' its fluctuating
274 part) and where $c_0^{\text{LBM}} = \sqrt{\gamma RT_0}$ is the lattice Boltzmann speed of sound owing to the acoustic scaling. Even
275 though this equation of state does not correspond to any physical fluid, it can be linked to more general
276 equations of state. Indeed, for small and nearly-isentropic disturbances any equation of state $p = p(\rho, s)$ can
277 be linearised as [54]:

$$p = p_0 + p' \approx p_0 + \rho' \left(\frac{\partial p}{\partial \rho} \right)_s = p_0 + \rho' c_0^2, \quad (18)$$

278 where $p_0 = \rho_0 RT_0$ and ρ_0 are the reference pressure and density, p' and ρ' the pressure and density deviations
279 w.r.t their reference value, and $c_0 = \sqrt{\gamma RT_0}$ the physical speed of sound. By comparing the fluctuating
280 parts of Equations (17) and (18), it can be seen that the LBM correctly computes the pressure fluctuations
281 even though the LBM reference pressure p_0^{LBM} does not correspond to the one in Equation (18) (they differ
282 by a factor of γ). Therefore, starting from the pressure fluctuations calculated by the LBM, it is proposed to
283 reconstruct the temperature fluctuations around its reference value T_0 by using perturbed perfect gas law:

$$T' = \frac{p' - \rho' RT_0}{(\rho_0 + \rho')R} = \frac{\rho' [(c_0^{\text{LBM}})^2 - RT_0]}{(\rho_0 + \rho')R}, \quad (19)$$

284 where R is the specific gas constant. To ensure the validity of the temperature reconstruction $T = T_0 + T'$
285 where T' is computed thanks to Equation (19), the coupling methodology presented in this paper imposes
286 some restrictions on the location of the interface between both solvers. As a matter of fact, it has to lie
287 in flow regions where the linearised Equation (18) is valid, i.e. in regions where entropy fluctuations are
288 negligible (e.g. in linear acoustics zones). To overcome this limitation, lattice Boltzmann methods including
289 thermal or ideal gas compressibility effects can be implemented (for instance by using multispeed lattices
290 [55], double distribution functions [56] or by coupling an extra energy equation, solved by finite differences,
291 to the LBM [57]). This will be the subject of future work.

292 3.4. Reconstruction of the distribution functions

293 With the results of Sections 3.2 and 3.3, a direct mapping from the distribution functions to the flow state
294 variables has been established. However, the spatial coupling introduced in this paper requires a two-way
295 exchange. Therefore, it is crucial to understand how the LB set of distribution functions can be derived
296 from the macroscopic flow state vector $\mathbf{W} = (\rho, u_i, T)^t$. Given its mesoscopic nature, the LBM genuinely
297 gathers more information than at the continuum level. Indeed, with the D3Q19 lattice, $q = 19$ distribution
298 functions have to be specified at the coupling interface arising from $D + 2 = 5$ flow variables computed by
299 the NS-FV solver. As a consequence, a one-to-many problem occurs. The same problem also appears when
300 initialising a LBM computation from macroscopic data [58].

301 Following the idea of Skordos [58], the vast majority of hybrid lattice Boltzmann - Navier-Stokes meth-
302 ods propose to split the distribution functions into an equilibrium and an off-equilibrium part. While the

303 equilibrium part can be directly computed thanks to its analytical formula, off-equilibrium distribution
304 functions can be determined by several manners. Albuquerque *et al.* [14] proposed to express them ana-
305 lytically through a Chapman-Enskog expansion. Such methodology has also been applied in similar ways
306 in [16, 59]. On the other side, Neumann *et al.* [19] solved a constrained-optimisation problem by ensuring
307 the mass, momentum and viscous stresses conservation at the interface in order to obtain the corresponding
308 off-equilibrium distribution functions. It should be noted that other methods using completely different
309 approaches were also developed such as constrained runs [59], velocity-boundary coupling [60] or statistical
310 inference [61].

311 The distribution functions reconstruction methodology proposed in this paper follows the fundamental
312 idea of Albuquerque *et al.* [14] relying on a Chapman-Enskog expansion and further extends it without any
313 prior assumption on the distribution functions. The hybrid method being designed for unsteady aerodynamic
314 and aeroacoustic application, solving an optimisation problem on each interface cell at each time step seems,
315 at first glance, quite expensive in terms of CPU time. Consequently, in the ghost cells of the LB solver, the
316 distribution functions will be given by (see Figure 2 for notations):

$$g_i(\square, t) = g_i^{eq}(\mathbf{U}(\circ, t)) + g_i^{neq}(\mathbf{U}(\circ, t)) \quad \text{for each } i \in \llbracket 1; 19 \rrbracket, \quad (20)$$

317 where the off-equilibrium component g_i^{neq} has to be determined. This is the topic of the present section
318 where a general methodology which can be applied to bridge the LBM with any other macroscopic model
319 is introduced.

320 3.4.1. Mixed Taylor/Chapman-Enskog expansion

321 As a first step towards the determination of the off-equilibrium component g_i^{neq} , a mixed Taylor/Chapman-
322 Enskog expansion is performed on the lattice Boltzmann scheme with the corrected HRR collision operator
323 (11) (see Appendix A for the detailed analysis). In all the previous work on hybrid lattice Boltzmann -
324 Navier-Stokes solvers [13, 14, 16, 18], this analysis was conducted by directly expanding the discrete dis-
325 tribution functions g_i around their equilibrium. However, as discussed in [62], there is a difference of one
326 order of magnitude in the expansion parameter between the continuous (f_i) and discrete (g_i) distribution
327 functions. As a consequence, in the analysis reported in Appendix A care has been taken to re-express the
328 discrete distributions (g_i) as a function of the continuous ones (f_i). The following relation is then obtained:

$$\left(\left[\frac{\partial}{\partial t_1} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}_1} \right] f_i^{eq} - \psi_i \right) = -\frac{1}{\tau} f_i^{neq,(1)} + \mathcal{O}(\Delta t^2). \quad (21)$$

329
330 This equation is the cornerstone of the coupling procedure. Indeed, it shows that the first-order truncated
331 off-equilibrium distribution functions $f_i^{neq,(1)}$ are completely determined by the knowledge of the equilibrium
332 distribution function and its associated space- and time-derivatives. Moreover, the first-order approximation

333 of f_i^{neq} has been shown to be sufficient to recover the macroscopic Navier–Stokes equations [37]. Therefore,
 334 the off-equilibrium contribution which has to be considered when reconstructing the distributions in the
 335 ghost cells of the LB solver is restricted to $f_i^{neq,(1)}$.

336 As highlighted by the $\mathcal{O}(\Delta t^2)$ error-term in Equation (21), the defining relation of $f_i^{neq,(1)}$ is second-order
 337 accurate which means that the second-order convergence of the LB scheme is naturally preserved.

338 3.4.2. Expressing the off-equilibrium contribution

339 Equation (21) clearly shows that the first-order approximation of the off-equilibrium distribution func-
 340 tions $f_i^{neq,(1)}$ is directly linked to gradients of the equilibrium distribution function f_i^{eq} . Yet, f_i^{eq} depends
 341 only on the macroscopic fields $\rho(\mathbf{x}, t)$ and $\rho\mathbf{u}(x, t)$, thus it becomes clear that $f_i^{neq,(1)}$ will likewise depend on
 342 the macroscopic fields and also on their gradients. This last point is of utmost importance as it brings in new
 343 information at the coupling interface and thus helps alleviating the one-to-many problem when reconstruct-
 344 ing the distribution functions from the macroscopic variables. Having said that, the analytical expression of
 345 $f_i^{neq,(1)}$ as a function of the macroscopic variables still has to be determined. In the present methodology,
 346 no prior approximations are made on the distribution functions f_i unlike in [13] where second-order terms
 347 in the equilibrium and both temporal and spatial derivatives of $f_i^{neq,(1)}$ were neglected.

348 The first step is to consider the equilibrium distribution function. Once the equilibrium is chosen, it can
 349 directly be replaced in Equation (21). Its time derivative, can then be simplified with the chain rule. As a
 350 consequence, time derivatives of macroscopic variables appear through:

$$\frac{\partial f_i^{eq}}{\partial t_1} = \frac{\partial f_i^{eq}}{\partial \rho} \frac{\partial \rho}{\partial t_1} + \frac{\partial f_i^{eq}}{\partial u_k} \frac{\partial u_k}{\partial t_1}, \quad (22)$$

351 which can also be expressed as spatial derivatives thanks to the macroscopic conservation equations. An
 352 analytical formulation of $f_i^{neq,(1)}$ is thereby obtained. Such procedure can be used for any equilibrium
 353 distribution function even if they differ from the one proposed in Equation (13).

354 It will now be shown that when using the hybrid recursive regularised collision operator for the LBM, the
 355 analytical expression of $f_i^{neq,(1)}$ is genuinely contained in the scheme. Indeed, in the present study, to ensure
 356 a smooth transition between the reconstruction interface and the LBM bulk solver, the same equilibrium
 357 as the one of Equation (13) is applied in the ghost cells of the LB solver. The latter relies on a Hermite
 358 expansion up to the third order [22]. By projecting Equation (21) onto the basis of Hermite polynomials, it
 359 simplifies into (see Appendix B for further details):

$$a_{1,\alpha_1 \dots \alpha_n}^{(n)} = u_{\alpha_n} a_{1,\alpha_1 \dots \alpha_{n-1}}^{(n-1)} + \sum_{i=1}^{n-1} u_{\alpha_1 \dots u_{\alpha_{n-2}}} a_{1,\alpha_i \alpha_n}^{(2)} \quad \text{for } n \geq 3, \quad (23)$$

360 where $a_1^{(n)}$ if n th-order Hermite coefficient of the off-equilibrium contribution $f_i^{neq,(1)}$, and $\alpha_1, \dots, \alpha_n$ are the
 361 tensor indices such as $\alpha_i \in \{x, y, z\}$ for each i . Equation (B.4), is almost the same as the one derived in

362 [49] however a slight difference appears implicitly in the definition of $\mathbf{a}_1^{(2)}$ where the corrective term helps to
 363 properly remove the $\mathcal{O}(\text{Ma}^3)$ error term which was neglected in [49] (see Appendix B for a short discussion).

364 In the context of the HRR-LBM and with Equation (B.4), the first-order approximation of the off-
 365 equilibrium part of the distribution functions $f_i^{neq,(1)}$ can be expressed as:

$$f_i^{neq,(1)} = w_i \sum_{n=0}^N \frac{1}{c_s^{2n} n!} \mathbf{a}_1^{(n)} : \mathcal{H}_i^{(n)}, \quad (24)$$

366 where the Hermite coefficient of the off-equilibrium distributions are given by:

$$\mathbf{a}_1^{(0)} = 0, \quad \mathbf{a}_1^{(1)} = 0, \quad \mathbf{a}_1^{(2)} = -\rho\tau c_s^2 (\nabla \mathbf{u} + \nabla \mathbf{u}^T), \quad a_{1,\alpha\beta\gamma}^{(3)} = u_\alpha a_{1,\beta\gamma}^{(2)} + u_\beta a_{1,\alpha\gamma}^{(2)} + u_\gamma a_{1,\alpha\beta}^{(2)}. \quad (25)$$

367 By comparing Equations (24) and (25) with Equation (13) and Table 1, it becomes clear that the analytical
 368 expression of $f_i^{neq,(1)}$ can be interpreted with regard to the one given by the HRR collision operator. Indeed,
 369 it is equivalent to applying the regularisation step in the ghost cells with a value of $\sigma = 0$ with the sole
 370 difference that all the macroscopic information is provided by the NS-FV solver. It has to be noticed that
 371 the choice of $\sigma = 0$ for the reconstruction of the off-equilibrium part introduces some numerical dissipation.
 372 However, this value is only applied in the ghost-cells of the LB solver thereby limiting its effect to a very
 373 small extent of the computational domain. As will be seen in Section 4, the numerical dissipation induced
 374 by $\sigma = 0$ in the ghost cells is found to be unnoticeable.

375 All in all, the missing distribution functions specified at the NS-FV to LBM interface are reconstructed
 376 by using the decomposition $f_i = f_i^{eq} + f_i^{neq,(1)}$ where f_i^{eq} and $f_i^{neq,(1)}$ are defined by Equations (13) and
 377 (24). The results obtained here with the space- and time-continuous distribution functions f_i have to be
 378 transposed to the discrete distribution functions g_i in order to apply Equation (20) in the LB ghost cells.
 379 As already stated in Section 2.2, $g_i^{eq} = f_i^{eq}$ in the framework of regularised collision operators. However,
 380 concerning the off-equilibrium part, the following relation has to be applied:

$$g_i^{neq,(1)} = \left(1 + \frac{\Delta t}{2\tau}\right) f_i^{neq,(1)} = \frac{\bar{\tau}}{\tau} f_i^{neq,(1)}, \quad (26)$$

381 meaning that the relations obtained previously remain valid in the discrete case provided that the continuous
 382 relaxation time τ is replaced by its discrete counterpart $\bar{\tau}$. The distribution functions $g_i(\square, t)$ (see Figure
 383 2 for notations) are then fully determined by the macroscopic variables and their gradients which will be
 384 computed with standard second-order centered finite differences hereafter.

385 3.5. Coupling of time advance schemes

386 Having seen how to spatially couple the LBM with the NS-FV solver, the last point to investigate
 387 concerns the coupling of time advance schemes. As already discussed in Section 2.2.1, the ‘‘Stream &
 388 Collide’’ algorithm of the LBM relies on a specific time and space discretisation leading to a second-order

389 explicit scheme. Moreover, the physical CFL number is imposed by the lattice constant c_s . As a consequence,
 390 the LB scheme in its classical form offers very little flexibility on the time integration scheme. By contrast,
 391 the NS-FV schemes are traditionally obtained through the method of lines meaning that both the spatial
 392 and temporal scheme can be chosen separately – provided that stability constraints are respected. Two
 393 time integration methods can be distinguished: the explicit and implicit ones. Explicit methods offer most
 394 of the time better accuracy and increased HPC capabilities. However, their time step is restricted by some
 395 stability criteria. Conversely, implicit methods allow bigger time steps due to their increased stability but
 396 at increased computational cost. For this reason, a coupling between the LBM and both explicit and implicit
 397 time schemes will be presented. Hereafter, it is assumed that both solvers use a common time step at least
 398 on each side of the coupling interface. The time step will be denoted by $\Delta t = \Delta t^{\text{NS}} = \Delta t^{\text{LBM}}$. Extending
 399 this methodology to non-conforming time-steps might be the purpose of future work.

400 3.5.1. LBM/Explicit coupling: the case of Runge-Kutta schemes

401 A s -step explicit Runge-Kutta (RK) method allows to compute \mathbf{U}^{n+1} – solution of Equation (1) – at
 402 time $t_{n+1} = t_n + \Delta t$ by using \mathbf{U}^n and $s - 1$ intermediate values. A s -step explicit Runge-Kutta method is
 403 defined as:

$$\begin{cases} t^{n,i} &= t_n + c_i \Delta t, \\ \mathbf{U}^{n,i} &= \mathbf{U}^n + \Delta t \sum_{j=1}^{i-1} a_{ij} \mathbf{R}^{n,j}, \\ \mathbf{R}^{n,i} &= \mathbf{R}(\mathbf{U}^{n,i}), \end{cases} \quad (27)$$

404 where $1 \leq i, j \leq s$. The i -th intermediate time, the i -th intermediate value, and the i -th intermediate
 405 evaluation of the residual between t_n and $t_n + \Delta t$ are denoted by $t^{n,i}$, $\mathbf{U}^{n,i}$, and $\mathbf{R}^{n,i}$ respectively. The
 406 solution at time t^{n+1} is then given by $\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \sum_{i=1}^s b_i \mathbf{R}^{n,i}$. The method is fully defined by its
 407 families of real coefficients a_{ij} , b_i , and c_i .

408 The key element to notice when trying to couple an explicit RK method with the LB scheme is that
 409 boundary conditions need to be specified for each intermediate time $t^{n,i}$. The solution in the LB domain at
 410 these instants does not exist since the evolution from t_n to $t_n + \Delta t$ is direct. Hence, the LB solution has to
 411 be interpolated for each intermediate time. To illustrate the time coupling procedure, Figure 3 details one
 412 time advance step in the case of the RK-3 scheme introduced in Section 2.1.

413 The RK-3 algorithm relies on the computation of 2 intermediate values thus, two intermediate LB
 414 solutions at $t_n + c_1 \Delta t$ and $t_n + c_2 \Delta t$ have to be interpolated in order to correctly apply the NS-FV 5-
 415 point stencil at the interface between both solvers. The interpolation is carried out by means of Lagrange
 416 polynomials. Given a set of $k + 1$ distinct data points $(t_j, \mathbf{U}_{\text{LBM}}^j)$, the k -th order Lagrange interpolating

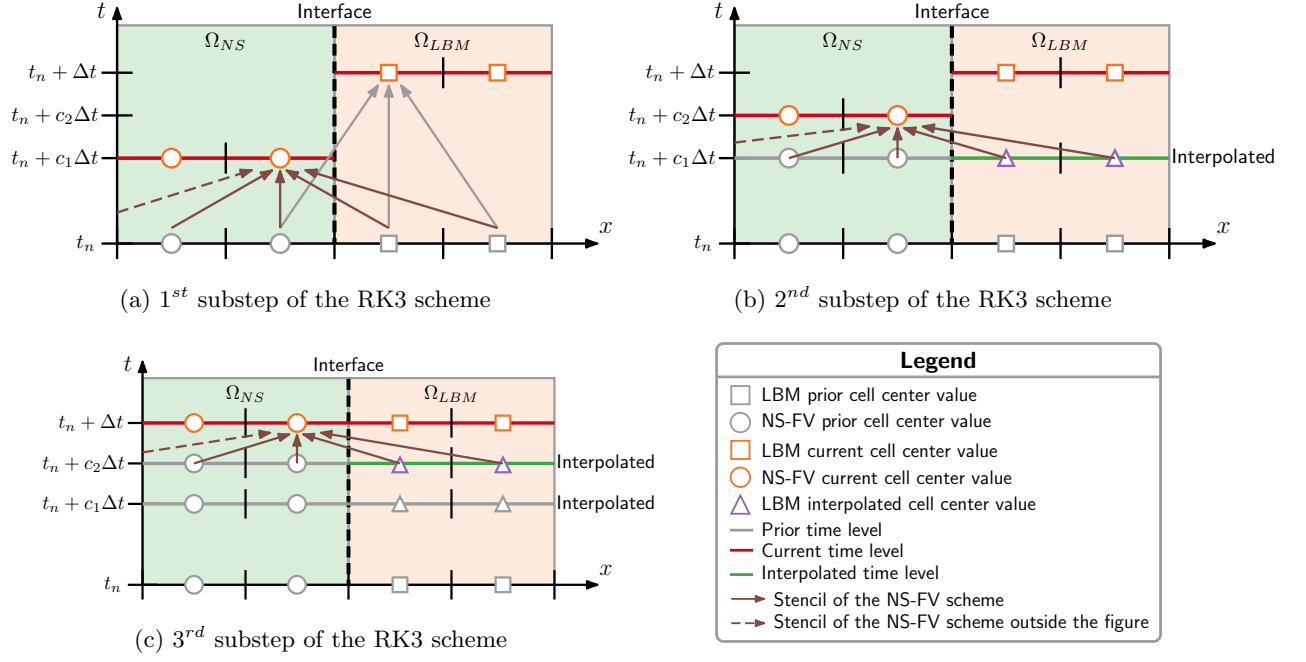


Figure 3: Temporal coupling between a 3rd order Runge Kutta scheme (RK3) and the lattice Boltzmann method.

417 polynomial is a linear combination of Lagrange basis polynomials:

$$\mathcal{L}(t) = \sum_{j=0}^k \mathbf{U}_{\text{LBM}}^j \left(\prod_{i=0, j \neq i}^k \frac{t - t_i}{t_j - t_i} \right). \quad (28)$$

418 In the following, 0th to 4th order interpolations will be compared. Table 2 summarizes the time levels
 419 and solution values used for each interpolation order. The interpolations are computed in a backward
 420 manner inasmuch as prior time levels are added to enrich the set of points needed to compute the Lagrange
 421 polynomial.

Order	$(t_{n-3}, \mathbf{U}_{\text{LBM}}^{n-3})$	$(t_{n-2}, \mathbf{U}_{\text{LBM}}^{n-2})$	$(t_{n-1}, \mathbf{U}_{\text{LBM}}^{n-1})$	$(t_n, \mathbf{U}_{\text{LBM}}^n)$	$(t_{n+1}, \mathbf{U}_{\text{LBM}}^{n+1})$
0	○	○	○	○	✓
1	○	○	○	✓	✓
2	○	○	✓	✓	✓
3	○	✓	✓	✓	✓
4	✓	✓	✓	✓	✓

Table 2: Data points required for the computation of the interpolating Lagrange polynomials. ○: solution not used, ✓: solution used.

422 3.5.2. LBM/Implicit coupling: the case of the Gear scheme

423 In some cases, implicit time stepping might be beneficial due to its ability to deal with larger time
424 steps. To this end, and to take advantage of the flexibility offered to users on choosing between various
425 time-stepping strategies for FV schemes, a coupling procedure between the LBM and an implicit scheme
426 is presented. More particularly, Gear's implicit scheme present in ONERA's FAST CFD suite [28] will be
427 used. Applied to Equation (1), it reads:

$$\mathcal{F}(\mathbf{U}^{n+1}) = 0, \quad \text{where} \quad \mathcal{F}(\mathbf{U}^{n+1}) = \frac{3\mathbf{U}^{n+1} - 4\mathbf{U}^n + \mathbf{U}^{n-1}}{2} + \frac{\Delta t}{|\Omega|} \mathbf{R}(\mathbf{U}^{n+1}). \quad (29)$$

428 The resulting non-linear problem is solved iteratively. At the beginning of each iteration the solution
429 increment is fixed at zero in the NS-FV ghost cells. This corresponds to a Dirichlet condition. Thus,
430 all boundary conditions are treated explicitly in our implementation, which has the advantage of being
431 computationally cheap. The role of the iterations of Newton's internal process is to remove the errors due
432 to the explicit treatment of the boundary conditions. As a result, the coupling of the LB scheme with this
433 implicit time stepping method is straightforward. The LB domains perform their iteration first then, the
434 solution at time t_{n+1} being known, it is applied on the boundaries of every NS domain throughout the entire
435 solving process of Equation (29).

436 4. Numerical results and hybrid method capabilities

437 In order to demonstrate the validity of the proposed hybrid lattice Boltzmann - Navier-Stokes method and
438 to highlight its capabilities when computing unsteady flow problems, some aerodynamic and aeroacoustic
439 test cases are conducted. First, the coupling components introduced in Section 3 are discussed on the case
440 of a 1D Gaussian acoustic wave. The coupling strategy is then validated with the case of a convected
441 vortex. In addition, the acoustic capabilities of the hybrid method are assessed through the computation
442 of the radiation of a harmonic acoustic source in a fluid medium at rest. Finally, the application of the
443 hybrid lattice Boltzmann - Navier-Stokes method to a representative aeronautical application where both
444 the aerodynamics and acoustics are required is introduced with the study of the flow past a circular cylinder.
445 The computational cost of the proposed hybrid method on this last test case will be discussed and compared
446 to full Navier-Stokes and lattice Boltzmann simulations.

447 All the simulations of the present study are carried out on pseudo-2D domains with a $10\Delta x$ transverse
448 extension where Δx is the grid step. Thus, the 3D formulation of the Boltzmann and Navier-Stokes lattice
449 solvers is used but periodic boundary conditions are used in the transverse direction so as to ensure that
450 the flow remains perfectly 2D. Moreover, the speed of sound is fixed at $c_0 = 347.3 \text{ m.s}^{-1}$, the specific gas
451 constant is given by $R = 287.053 \text{ J.kg}^{-1}.\text{K}^{-1}$ and the heat capacity ratio is $\gamma = 1.4$.

452 *4.1. Gaussian acoustic plane wave*

453 The first test case introduced here is a convected one dimensional acoustic wave. Besides validating
 454 the lattice Boltzmann - Navier-Stokes coupling strategy, this simple problem will also help to highlight the
 455 influence of the different coupling components introduced in Section 3.

456 The computational domain, which is represented in Figure 4, is a periodic box of size $[2L, L, 10\Delta x]$
 457 decomposed in 2 sub-domains Ω_{NS} and Ω_{LBM} of size $[L, L, 10\Delta x]$ each, with L being the reference length
 458 equal to 1 m. Periodic boundary conditions are implemented at the borders of the computational domain in
 459 the $x, y,$ and z directions. A 1D Gaussian acoustic plane wave is initialised at the center of the NS domain
 460 and propagates towards the LB domain thus crossing the coupling interface.

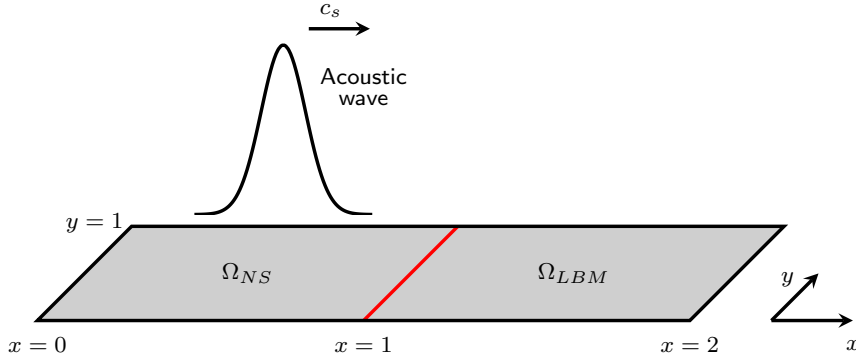


Figure 4: Schematic representation of the 1D acoustic plane wave test case.

461 At the beginning of the computation, the flow variables are defined as follows:

$$\begin{cases} \rho(x) &= \rho_0 \left(1 + A \exp \left(\frac{-(x-x_c)^2}{2R_c^2} \right) \right), \\ u_x(x) &= U_x - \frac{\rho'}{\rho_0} c_0, \end{cases} \quad (30)$$

462 where $\rho_0 = 1.1765 \text{ kg.m}^{-3}$ is the free-field density, $A = 10^{-4}$ is the amplitude of the density perturbation,
 463 and $x_c = 0.5$ is the initial location of the wave. The width of the Gaussian is controlled by $R_c = 20\Delta x$
 464 ensuring a well resolved wave. Moreover, a mean flow U_x directed along the positive x -axis has been added
 465 corresponding to a Mach number of 0.1. Only the case of a wave propagating from the NS domain to the
 466 LBM domain is presented here as the results are the same for a wave propagating in the opposite direction.

467 The uniform grid size is set to $\Delta x = L/200$ and the time-step is chosen so as to enforce a CFL number
 468 based on the upstream velocity $\text{CFL} = 1/\sqrt{3} \approx 0.57$ for both the NS-FV and lattice Boltzmann solvers.
 469 Time integration on the NS-FV side will be performed with the 3rd-order explicit Runge-Kutta scheme.

470 First of all, the influence of the thermodynamic closure derived in Section 3.3 is investigated. Two
 471 simulations are run: one where a constant temperature T_0 is specified at the NS-FV solver on the interface
 472 and another one taking into account the temperature fluctuations T' according to Equation (19). Indeed,

473 such reconstruction is valid in the present context since the temperature field in the NS solver has been
 474 initially computed according to the Laplace law for isentropic flows. Moreover, for air (i.e. with low viscosity
 475 and thermal conductivity) the entropy production can be neglected when propagating the wave over small
 476 distances of the order of L . Both computations are run until the acoustic wave reaches the interface between
 477 both solvers. Figure 5 compares the corresponding relative density profiles $(\rho - \rho_0)/A$. It can be seen that
 478 when applying a constant temperature, spurious oscillations appear in the density and temperature fields
 479 on the Navier-Stokes side. By contrast, the lattice Boltzmann solver is unaffected by these oscillations since
 480 no equation for energy conservation is solved with a D3Q19 lattice. On the other hand, thanks to the
 481 temperature fluctuations estimation presented in Section 3.3, a smooth transition is recovered. Indeed, the
 482 density profile remains continuous across the interface between both numerical methods and the solution
 483 perfectly matches the analytical one. Therefore, only computations taking into account the reconstructed
 484 temperature fluctuations will be shown hereafter.

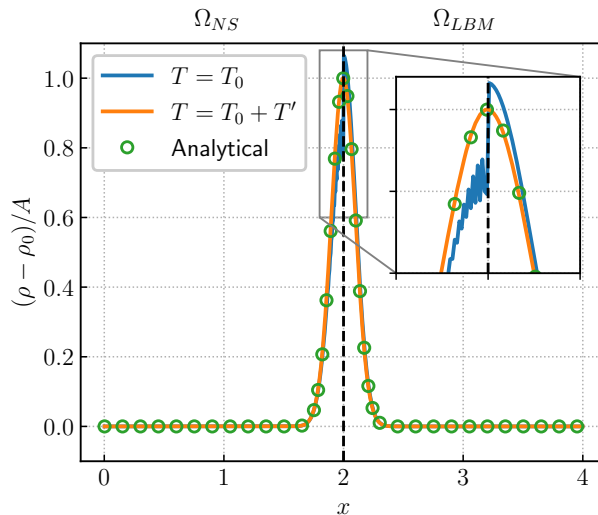


Figure 5: Relative density profile $(\rho - \rho_0)/A$ for the 1D convected acoustic wave test case. Two computations are compared: one where a constant temperature T_0 is specified at the NS-FV solver on the interface and another one taking into account the temperature fluctuations T' according to Equation (19).

485 The reconstruction of distribution functions is now considered. Again, two simulations are run: one
 486 where the LB distribution functions are specified to their equilibrium value at the interface and another
 487 one taking into account the off-equilibrium contribution $g_i^{neq,(1)}$ given by Equation (24). As previously, the
 488 computations are run until the acoustic wave reaches the interface between both solvers. This comparison
 489 has been motivated by previous results [14, 59] where the authors pointed out the importance of considering
 490 the off-equilibrium part. In Figure 6 the corresponding relative density profile and the density gradient
 491 are shown. While in Figure 6a both reconstructions seem to lead to a satisfying result matching the

492 analytical solution, further analysis highlights some discrepancies. Indeed, the density gradient exhibits a
 493 non-smooth behaviour in the vicinity of the coupling interface when distribution functions are reconstructed
 494 using only their equilibrium value. Over time, i.e. after many crossings of the interface, the waveform is
 495 eventually strongly degraded. In comparison, taking into account more macroscopic information through
 496 the off-equilibrium contribution $g_i^{neq,(1)}$ enables to recover a smooth gradient which perfectly matches the
 497 analytical one. As a consequence, one can conclude that the off-equilibrium contribution is necessary to
 498 obtain the correct profiles and thus the correct acoustics. This result, largely demonstrated in the stationary
 499 case [14, 59], is therefore also verified in the unsteady case. As a consequence, the off-equilibrium contribution
 500 is always applied in the following computations.

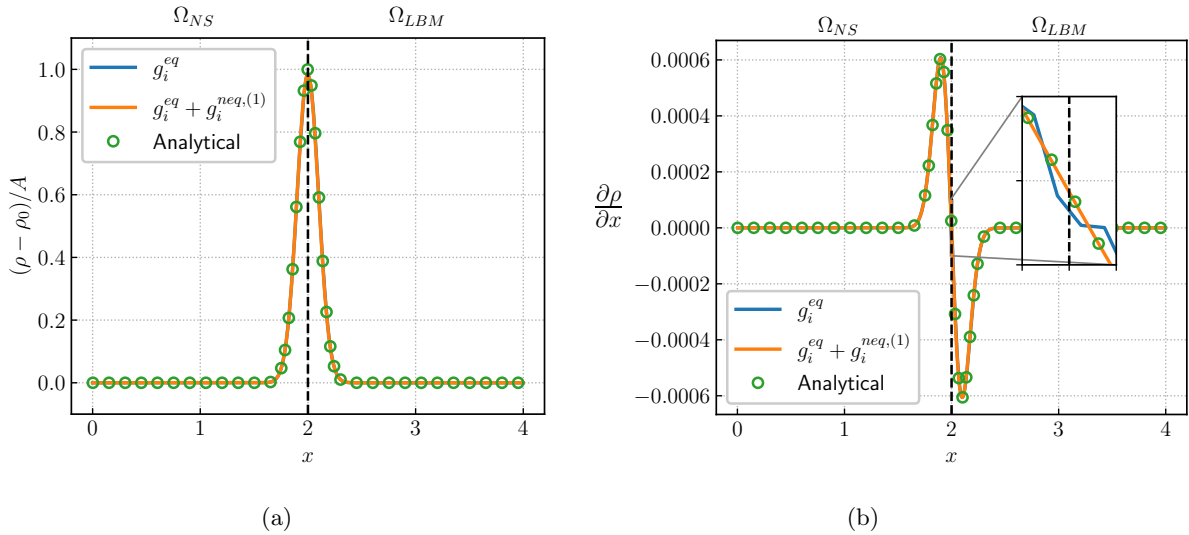


Figure 6: (a) Relative density profile $(\rho - \rho_0)/A$ and (b) Density gradient for the 1D convected acoustic wave test case. Two computations are compared: one where the LB distribution functions are specified to their equilibrium value at the interface and another one taking into account the off-equilibrium contribution.

501 To complete this discussion on the different coupling components introduced in Section 3, the influence
 502 of the time interpolation and of the order of the Lagrange interpolating polynomials is now studied. Figure
 503 7a displays the relative density profile $(\rho - \rho_0)/A$ after the first interface crossing for all the interpolation
 504 orders considered. Regardless of the choice made concerning the order of the Lagrange polynomial, the
 505 solution compares well with the analytical solution. However, some spurious acoustic waves, highlighted by
 506 a rectangular box, are reflected at the coupling interface and travel upstream. Figure 7b provides a closer
 507 look to them. First thing to notice is that the amplitude of these spurious waves represent less than 0.1%
 508 of the incident physical wave. More interestingly, this figure helps to characterise the overall effect of the
 509 interpolation order: by increasing the set of points used for the interpolation – and thus the order of the
 510 Lagrange polynomial – the amplitude of the spurious reflected waves is significantly reduced. Since both

511 the third- and fourth-order interpolation lead to similar results, one can conclude that there is no need to
 512 employ higher-order polynomials. Using a 3rd-order Lagrange polynomial interpolation eventually leads to
 513 a reflected wave representing less than 0.025% of the amplitude of the incoming wave which is considered
 514 as being acceptable (Figure 8 shows their negligible effect on the overall computation after 10 advection
 515 cycles). The interpolation process having converged, the remaining spurious acoustics are expected to arise
 516 from the change in numerical methods. As a complement, Figure 7c presents the evolution of the L_2
 517 density error integrated over the whole computational domain as a function of the iteration number. The
 518 two vertical dashed lines represent the instant when the pulse starts (and respectively ends) crossing the
 519 coupling interface. On this Figure, two kind of errors are noticeable. The first one corresponds to the
 520 numerical error of each scheme. Indeed, before the first dashed line, the L_2 error constantly increases owing
 521 to the numerical dissipation of the Navier-Stokes solver. On the other hand, and in agreement with the
 522 results of [7], the LB solver exhibits a lower numerical dissipation of the acoustic wave since the L_2 error
 523 tends to remain constant over time. The other error component which is evidenced between the two dashed
 524 lines is to one arising from the crossing of the acoustic pulse through the coupling interface. As one might
 525 expect in the light of the previous Figures, increasing the interpolation order tends to smooth-out the error
 526 peaks which clearly indicate the presence of spurious emissions.

527 More interestingly, the order of the Lagrange interpolating polynomials fully conditions the error level
 528 after the coupling interface crossing, and by increasing the latter, the L_2 density error is diminished. Once
 529 again, the results are more or less the same whether we take an interpolation of order 3 or 4. As consequence,
 530 only 3rd-order temporal interpolations will be considered in the following as the spurious acoustics can be
 531 seen as negligible in comparison with the physical acoustic phenomenon simulated.

532 In order to get a more quantitative insight into the influence of each of the coupling components, Table
 533 3 provides the values of the L^2 density error after one interface crossing (i.e. when the acoustic wave
 534 reaches the center of the LBM domain). In Table 3, the first two columns indicate whether the temperature
 535 reconstruction or the off-equilibrium contribution of the distribution functions are taken into account (✓)
 536 or not (✗) whereas the other columns refer to the order of the time interpolation. The L^2 density error is
 537 defined as:

$$L^2(\rho) = \sqrt{\sum_{N_x, N_y, N_z} [\rho(x, y, z) - \rho_{\text{ana}}(x, y, z)]}, \quad (31)$$

538 where N_x , N_y , and N_z are the number of grid points in the computational domain along the x , y , and z axes
 539 respectively. The analytical density field is denoted by ρ_{ana} .

540 By comparing the first two lines of Table 3 with the last two ones, it can be seen that it is crucial to
 541 reconstruct the temperature fluctuations at the coupling interface. Indeed, regardless of the order of the
 542 time interpolation, the computations where only the reference temperature T_0 is imposed at the coupling
 543 interface exhibit an error larger of one order of magnitude compared to the ones where the temperature

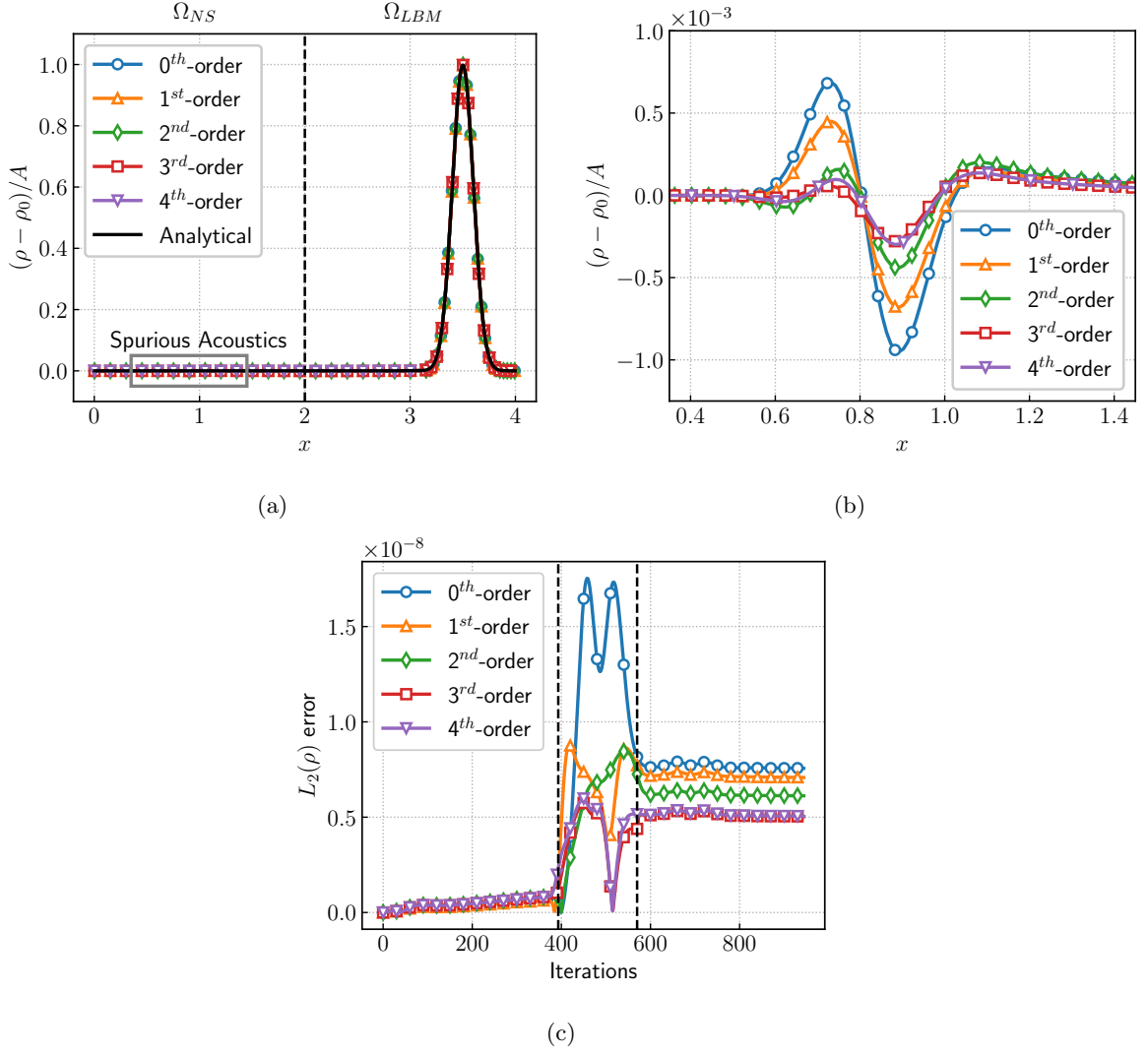


Figure 7: Influence of the interpolation order on spurious acoustics. (a) Relative density profile, (b) Zoom on spurious acoustic waves, (c) L_2 error as a function of iteration number.

T'	$g_i^{neg,(1)}$	0 th -order	1 st -order	2 nd -order	3 rd -order	4 th -order
\times	\times	3.189×10^{-7}	3.132×10^{-7}	3.068×10^{-7}	2.950×10^{-7}	2.898×10^{-7}
\times	\checkmark	3.180×10^{-7}	3.093×10^{-7}	2.982×10^{-7}	2.858×10^{-7}	2.803×10^{-7}
\checkmark	\times	1.097×10^{-8}	1.073×10^{-8}	1.041×10^{-8}	1.019×10^{-8}	0.984×10^{-8}
\checkmark	\checkmark	7.563×10^{-9}	7.089×10^{-9}	6.116×10^{-9}	5.046×10^{-9}	5.039×10^{-9}

Table 3: L^2 density error after one interface crossing of the acoustic wave. The effects of the temperature fluctuations reconstruction, the off-equilibrium contribution and the time interpolation are investigated.

544 fluctuations are reconstructed through Equation (19). In addition, comparing the first two lines of Table 3
545 shows that the error due to an incorrect thermodynamic closure is prevalent since adding the off-equilibrium
546 contribution to the distribution functions has little effect on the level of the L^2 error. Consequently, the
547 positive effect of taking into account the off-equilibrium contribution in the distribution functions is only
548 highlighted once the thermodynamic closure is properly addressed. This is shown in the third line of Table 3
549 where a twofold reduction in the L^2 error is observed when the off-equilibrium is taken into account. Finally,
550 Table 3 also confirms the results of Figure 7: increasing the time interpolation order helps in reducing the
551 error level at the end of the computation. It can be noted that the L^2 errors in the last line of Table 3 are
552 exactly the same as the ones shown in Figure 7c.

553 To conclude the analysis of this test case, the stability and robustness of the coupling procedure are
554 discussed by performing the computation over 10 advection cycles. As shown by the last line of Table 3, the
555 use of a fourth-order time interpolation does not lead to a significant reduction in error compared to third
556 order interpolation. Therefore, only 0th- to 3rd-order time interpolations are considered in the following.

557 Table 4 provides the values of the L^2 density error after 10 advection cycles. Overall, the conclusions
558 reached in the study of Table 3 are also confirmed after 10 advection cycles. It is important to note that
559 the value of the temperature correction introduced in Section 3.3 is exacerbated after 10 advection cycles
560 insofar as the error is two orders of magnitude greater in the cases where this correction is not taken into
561 account. Besides, the last line of Table 4 demonstrates the robustness of the coupling procedure when taking
562 into consideration the temperature fluctuations and the off-equilibrium contribution reconstructions at the
563 interface between both methods. Indeed, while the error values are slightly higher than in Table 3 (mainly
564 due to dissipation as shown by Figure 8), they seem to remain bounded which proves that the coupling error
565 does not lead to an accumulation that could compromise the stability and quality of the solution.

T'	$g_i^{neq.(1)}$	0 th -order	1 st -order	2 nd -order	3 rd -order
✗	✗	5.399×10^{-6}	5.384×10^{-6}	5.352×10^{-6}	5.321×10^{-6}
✗	✓	5.288×10^{-6}	5.272×10^{-6}	5.238×10^{-6}	5.206×10^{-6}
✓	✗	6.562×10^{-8}	5.498×10^{-8}	4.1630×10^{-8}	3.246×10^{-8}
✓	✓	3.063×10^{-8}	2.853×10^{-8}	2.369×10^{-8}	1.892×10^{-8}

Table 4: L^2 density error after 10 advection cycles of the acoustic wave.

566 To substantiate these statements, Figure 8 shows the corresponding relative density profile and the
567 density gradient after 10 advection cycles when both the temperature fluctuations and the off-equilibrium
568 contributions reconstructions are considered. It can be seen that, even after many crossings of the interface,
569 the waveform is preserved and perfectly matches the analytical profile. Furthermore, the spurious acoustic

570 waves induced by the coupling interface do not build up and remain unnoticeable at the scale of the initial
571 perturbation. Moreover, Figure 8b focuses on the density gradient. Thanks to the off-equilibrium contri-
572 bution $g_i^{neq,(1)}$, the gradient is also in good agreement with the its analytical counterpart and very limited
573 spurious oscillations are exhibited. In addition to the computation over 10 advection cycles, another one
574 over 100 advection cycles is also performed and the amplitude of the spurious waves is still found to be
575 negligible (of the order of 1% of the acoustic wave). Consequently, it can be concluded that the coupling
576 strategy remains stable despite the minor errors generated at the interface between both numerical methods.

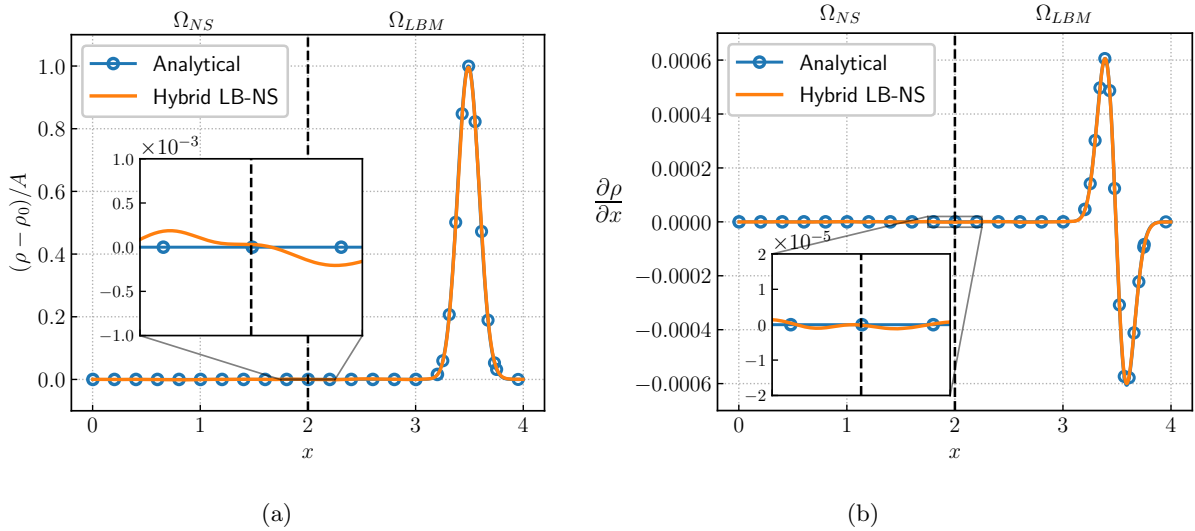


Figure 8: (a) Relative density profile $(\rho - \rho_0)/A$ and (b) Density gradient for the 1D convected acoustic wave test case after 10 advection cycles. In the present computation, both the temperature fluctuations and the off-equilibrium contributions are used. In addition, a third-order time interpolation is employed.

577 4.2. Convected vortex

578 The convected vortex is a classic Euler benchmark which is often used in the context of computational
579 fluid dynamics to assess numerical schemes in terms of dispersion, dissipation, and robustness. As discussed
580 in dedicated publications [62, 63], care has to be taken at the initialisation stage when using the lattice
581 Boltzmann method. Indeed, the classical isentropic formulation of the convected vortex test case is not con-
582 sistent with the athermal approximation of standard LBMs since in such approaches no energy conservation
583 equation is solved. An improper initial field can then lead to strong spurious waves compromising the entire
584 computation. To alleviate this issue, a more suited “barotropic” version of the widely used Taylor vortex

585 derived in [63] is chosen. It reads:

$$\begin{cases} \rho(x, y) &= \rho_0 \exp \left[-\frac{\epsilon^2}{2c_s^2} \exp \left(\frac{-r^2}{R_c^2} \right) \right], \\ u_x(x, y) &= U_0 - \epsilon \left(\frac{y - y_c}{R_c} \right) \exp \left[-\frac{(x - x_c)^2 + (y - y_c)^2}{2R_c^2} \right], \\ u_y(x, y) &= \epsilon \left(\frac{x - x_c}{R_c} \right) \exp \left[-\frac{(x - x_c)^2 + (y - y_c)^2}{2R_c^2} \right], \end{cases} \quad (32)$$

586 where $\rho_0 = 1.1765 \text{ kg}\cdot\text{m}^{-3}$ is the free-stream density, $U_x = 0.1c_0$ is the advection speed, $\epsilon = 0.07c_0$ is the
 587 vortex strength, and $R_c = 0.1 \text{ m}$ is the characteristic radius of the vortex. The center of the vortex is
 588 initially positioned at (x_c, y_c) . The corresponding analytical solution is given by the exact same pattern
 589 simply advected by the mean flow over time.

590 The computational domain, which is shown in Figure 9, is a box of size $[3L, L, 10\Delta x]$ decomposed in
 591 3 sub-domains of size $[L, L, 10\Delta x]$ each, with L being the reference length equal to 1 m. Two NS sub-
 592 domains, located at each side of a LBM sub-domain, enable to study the two-way crossing of vortical
 593 structures. Periodic boundary conditions are implemented at the borders of the computational domain in
 594 all three directions of space.

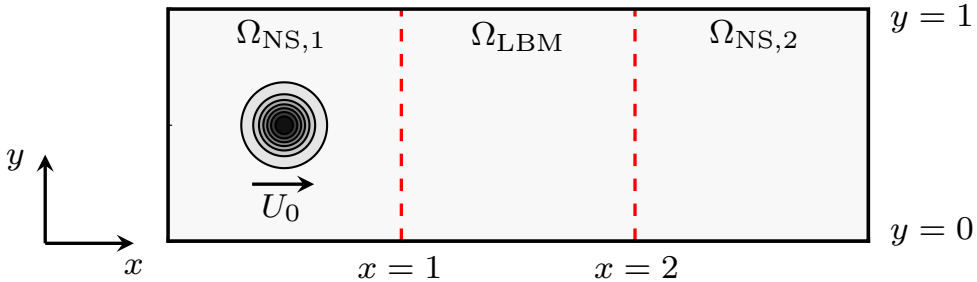


Figure 9: Schematic representation of the computational domain for the convected vortex test case.

595 The uniform grid size is set to $\Delta x = L/N$ where N is the number of grid points per unit length and the
 596 time-step is chosen so as to enforce a CFL number based on the upstream velocity $\text{CFL} = 1/\sqrt{3} \approx 0.57$ for
 597 both the NS-FV and lattice Boltzmann solvers thus ensuring a synchronous evolution. In the following, the
 598 grid resolution is fixed to $N = 200$ unless otherwise stated leading to 20 cells in the radius of the vortical
 599 structure. Moreover, this test case is performed in the inviscid limit to get rid of the viscous dissipation and
 600 directly investigate the numerical dissipation of the hybrid method. This also allows to study the stability
 601 of the hybrid method. As a consequence, only convective fluxes will be evaluated by the FV solver and the
 602 LB relaxation time is set to $\bar{\tau} = 0.5$.

603 First, the results obtained with the NS-FV method equipped with the 3rd-order explicit Runge-Kutta
 604 scheme are presented. Figure 10 shows the relative density and velocity profiles for the first (Figure 10a)

605 and tenth (Figure 10b) advection cycles respectively at five different stages. One should notice that, one
 606 advection cycle refers to the time needed for the vortex to be advected back to its initial position.

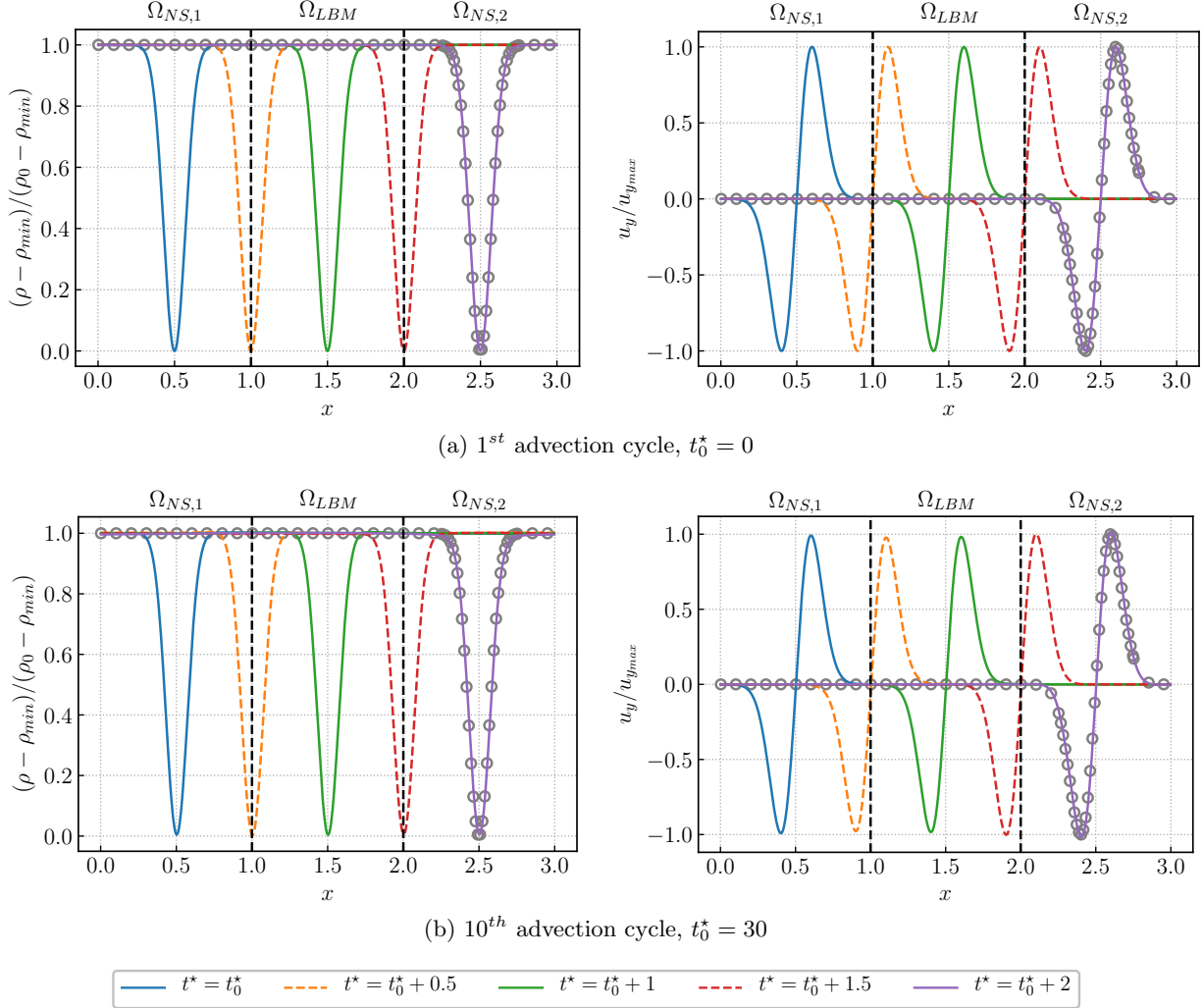


Figure 10: Density (left) and u_y velocity component (right) profiles for the periodic convected vortex for the (a) first and (b) tenth advection cycle. Time is expressed through its non-dimensional value $t^* = t/T_{\text{covo}}$ where $T_{\text{covo}} = N/U_0$. The analytical solution (\bigcirc) is superimposed at $t^* = t_0^* + 2$.

607 It can be observed from Figure 10 that the vortex crosses the two coupling interfaces without any
 608 distortion: the shape of the vortex is preserved regardless of the number of advection cycles and no spurious
 609 oscillations are visible at the transitions between both solvers. Moreover, analytical profiles (indicated by
 610 circles \bigcirc) have been superimposed on Figure 10 and show that all the results are in good agreement with
 611 the analytical solution.

612 The numerical dissipation of the hybrid method is now quantified with the following parameter $\xi =$
 613 $\frac{\min \rho(t_0) - \min \rho(t_f)}{1 - \min \rho(t_0)}$. A value of $\xi = 8 \times 10^{-5}$ is obtained after 10 advection cycles meaning that less than 0.01%

614 of the initial amplitude is lost. Furthermore, the numerical dissipation is found to be $\xi = 3 \times 10^{-4} < 0.05\%$
615 after 30 advection cycles. As a consequence, the hybrid method has very little intrinsic numerical dissipation
616 thanks to the combination of the hybrid centered/decentered convective fluxes in the NS-FV solver and the
617 value of $\sigma = 0.995$ which is used for the HRR collision operator. This also indicates that the value of
618 $\sigma = 0$ which is imposed in the LB ghost-cells by the coupling procedure has a negligible effect on the overall
619 dissipation of the hybrid method. Note that a smaller value of σ can be chosen in the bulk solver to increase
620 the stability however owing to the fact that computations remain stable after 50 advection cycles, it is
621 concluded that the value of $\sigma = 0.995$ is sufficient to ensure stable results.

622 In order to get more insight into the results obtained with the hybrid lattice Boltzmann - Navier-Stokes
623 method, the derivatives of the flow variables are computed. As indicated in [15], when coupling two numerical
624 methods, the quality of the vorticity field is more sensitive than the velocity field at the interface. Indeed,
625 vorticity, which is defined by $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, involves first-order derivatives of the velocity field. Consequently,
626 ensuring its smoothness requires stricter conditions on the velocity field. To investigate this particular point,
627 Figure 11 displays the vorticity field in the vicinity of the first coupling interface (i.e. the one between $\Omega_{NS,1}$
628 and Ω_{LBM}) after 10 advection cycles.

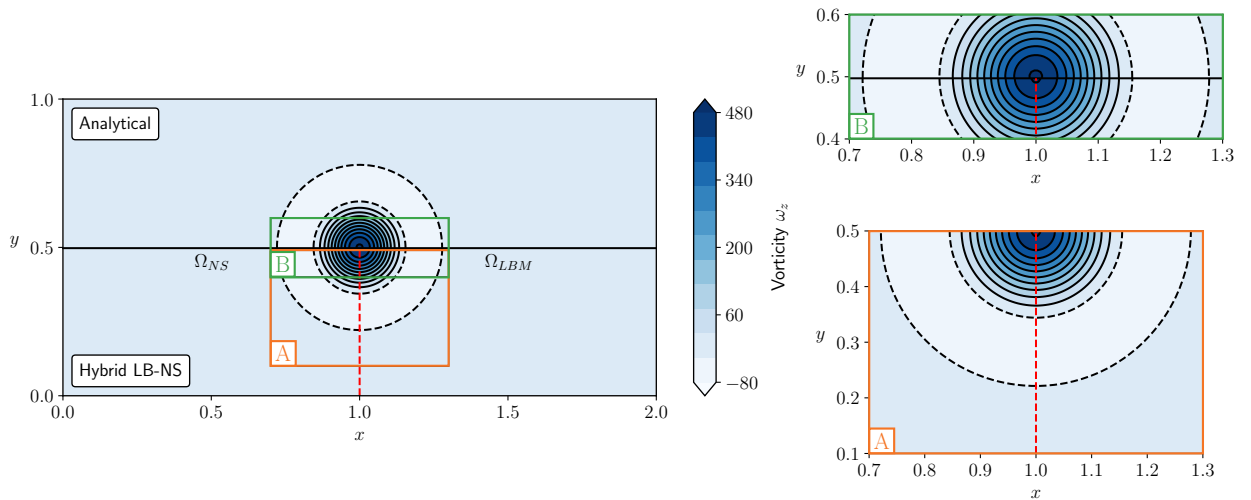


Figure 11: Vorticity field in the vicinity of the first coupling interface (centered at location $x = 1$). Iso-contours of vorticity are displayed. The contour levels are from $\omega_{min} = -80$ to $\omega_{max} = 480$ with an increment of 46. — : $\omega > 0$ and - - - : $\omega < 0$.

629 The vorticity field shown on Figure 11 is split in two: the upper half corresponds to the analytical vorticity
630 field while the lower half represents the one computed by the hybrid method. Moreover, 12 iso-contours
631 evenly spaced between $\omega_z = -80 \text{ s}^{-1}$ and $\omega_z = 480 \text{ s}^{-1}$ have been superimposed to the vorticity fields. The
632 results indicate that the isotropy of the solution is preserved. In addition, all contours are continuous and
633 do not exhibit any oscillations nor abrupt slope changes. Consequently, one can conclude that the proposed

634 coupling strategy not only guarantees continuous velocity fields but also a continuous vorticity field across
 635 the interface between both solvers.

636 Different cases with the mean flow spanning all three axis major directions and diagonals have been tested
 637 and have shown similar results to those of Figure 10 and Figure 11 thereby validating the 3D implementation
 638 of the hybrid method.

639 As the coupling strategy is validated with an explicit time advance scheme for the NS-FV side, the
 640 coupling between the LBM with a NS-FV solver relying on an implicit time scheme is now analysed. The
 641 exact same computational domain as the one presented in Figure 9 is considered. Both the grid size and the
 642 time step (and thus the CFL number) remain unchanged. The aim is not only to validate the LBM/implicit
 643 coupling but also to characterise the effect of the numerical errors induced by the explicit treatment of
 644 the boundary conditions (see Section 3.5.2). The computations are therefore compared to their hybrid
 645 LBM - NS-FV explicit counterpart so as to only highlight the effect of changing the time-marching method.
 646 As stated the implicit scheme involves Newton sub-iterations. Therefore, the influence of the convergence
 647 criteria is also investigated. .

648 On Figure 12, the relative density profile and the error $\rho(x, y) - \rho_{\text{ref}}(x, y)$ at $t^* = 4.5$ are plotted.
 649 The computation is stopped when the vortex crosses the $\Omega_{\text{LBM}} \rightarrow \Omega_{\text{NS},2}$ interface to clearly highlight
 650 the boundary treatment in the LBM/implicit time coupling. Three values of sub-iterations number are
 651 considered namely 3, 7, and 20. These, equivalently, correspond to an L_∞ norm of the Newton residual ϵ of
 652 10^{-1} , $10^{-2.2}$, and 10^{-5} respectively.

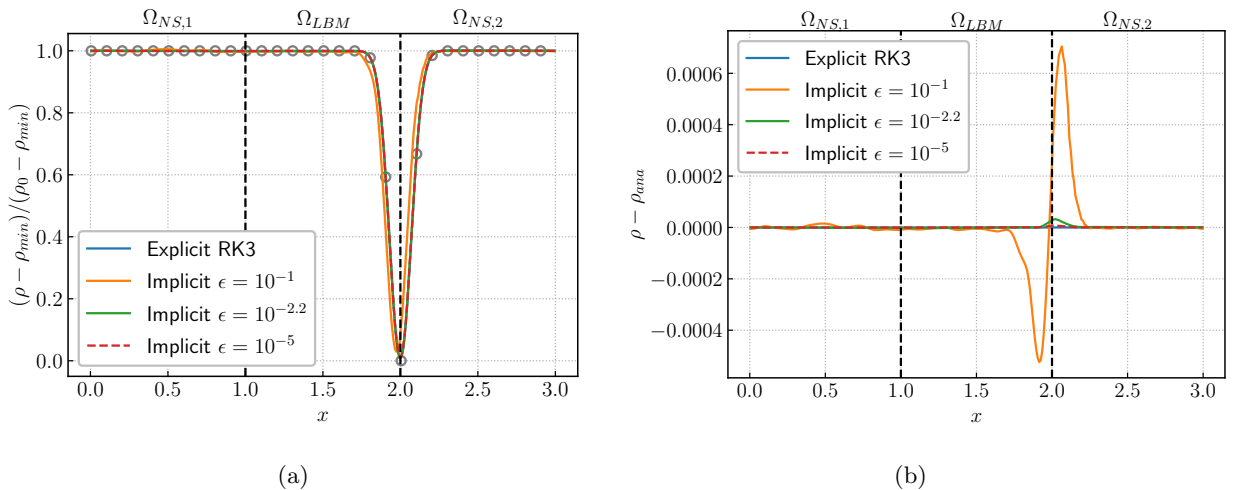


Figure 12: Density (a) and density error (b) profiles for the periodic convected vortex with an implicit time scheme for the NS-FV solver. The analytical solution (O) is superimposed on (a).

653 As evidenced by Figure 12a, an insufficient number of sub-iterations ($\epsilon = 10^{-1}$) leads to significant

654 numerical dissipation and dispersion errors. In fact, the error which is reported on Figure 12b reflects this
655 phenomenon. Naturally, decreasing the value of ϵ tends to improve the solution. The parameter ϵ has to
656 be small enough to damp out the spurious effect of the explicit treatment of the boundary conditions in
657 the Newton process. For the convected vortex test case, $\epsilon = 10^{-5}$ seems to be sufficient as the solution is
658 almost perfectly superimposed on the explicit and the analytical ones (indicated by circles \circ). Convergence
659 to machine accuracy is not required. Moreover, as in the explicit case, the vorticity remains continuous at
660 the interface between the NS-FV and LBM solvers when using the implicit time stepping.

661 Yet, one has to notice that on this particular test case, in order to achieve the same level of accuracy, the
662 implicit method is less computational efficient than the explicit one. However, as will be seen in Section 4.4,
663 using an implicit time integration scheme for the NS-FV solver can be beneficial when doing computations
664 on domains with a large disparity of grid sizes.

665 The convergence order of the hybrid lattice Boltzmann - Navier-Stokes method on the convected vortex
666 test case, is now determined. Knowing the exact solution at any given time, the error between the simulation
667 result and the translated initial conditions can be easily computed. For any flow quantity q , the relative L^2
668 error is defined as:

$$L^2(q) = \sqrt{\frac{1}{N_x N_y N_z} \frac{\sum_{x,y,z} [q(x,y,z) - q_{ref}(x,y,z)]^2}{\sum_{x,y,z} q_{ref}(x,y,z)^2}}, \quad (33)$$

669 where N_x , N_y , and N_z are the number of grid points in the computational domain along the x , y , and z axes
670 respectively. The exact value of the flow quantity q is denoted by q_{ref} .

671 The convected vortex test case is conducted on the computational domain of Figure 9 for a range of grid
672 resolution N with $N \in \{25, 50, 100, 200, 400\}$. For each value of N , the vortex is advected for 10 cycles and
673 both the density and velocity L^2 errors are computed. Figure 13 shows the x -velocity L^2 error versus the
674 vortex resolution (i.e. the number of grid points in the radius of the vortical structure $R_c = N/10$) for the
675 hybrid method as well as for a full Navier-Stokes and a full LBM computation using the exact same grid
676 and time steps. A second-order reference slope is also added to the plot to ease the interpretation of the
677 results.

678 The first thing to notice is that the classical second-order convergence of the LBM [64] is retrieved. In
679 comparison, the Navier-Stokes Finite-Volume solver seems to have a spatial order higher than 2. This is due
680 to the fact that all the computations are run using the time-step defined by Equation (9). Thus, changing
681 the vortex resolution also affects the time step. As a consequence, the slope obtained by the NS solver is
682 greater than 2 owing to the 3rd order convergence rate of the explicit Runge-Kutta scheme. When using the
683 implicit Gear scheme, which is second order in time, a slope of 2 is recovered. This being said, the general
684 conclusion of [7] is also recovered: at low resolutions ($N = 25$ or $N = 50$), the lattice Boltzmann method is
685 more precise than the NS-FV method because of its low numerical dissipation.

686 Moving now to the analysis of the hybrid method, one can see that for all vortex resolutions, the L^2

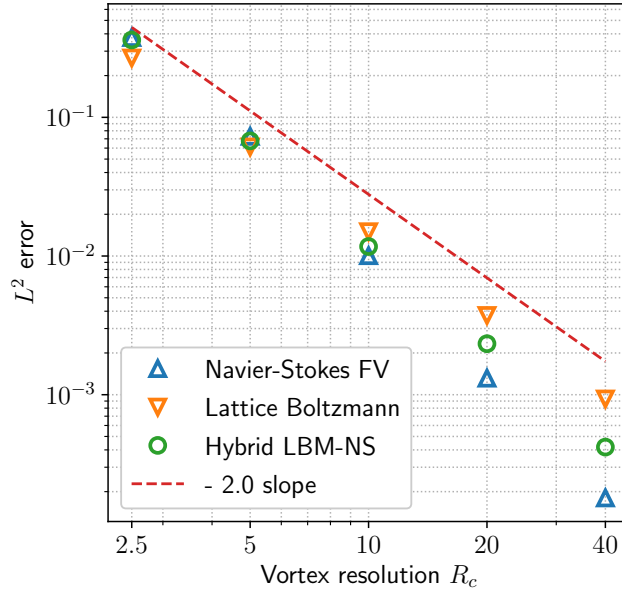


Figure 13: Convergence rate study of the proposed hybrid lattice Boltzmann - Navier-Stokes method. The relative error is computed through L^2 error on the x -velocity component after 10 advection cycles.

687 error always lies between the one of the LB solver and the one of the NS-FV solver. By comparison with the
688 reference second-order slope, it becomes clear that the coupled method has a second order convergence rate.
689 Notwithstanding the fact that only the x -velocity L^2 error is represented on Figure 13, the same results are
690 obtained for all the others flow quantities. This result is in accordance with previous studies [13, 59].

691 4.3. Harmonic acoustic source radiation in a fluid medium at rest

692 The hybrid lattice Boltzmann - Navier-Stokes method is now assessed on an acoustic propagation prob-
693 lem. This constitutes a first step towards the computation of realistic cases where the source is resolved with
694 one method whereas the other one is used to propagate the acoustic waves. Indeed, it is of paramount
695 importance to evaluate the acoustic capabilities of a numerical scheme in the field of computational aero-
696 acoustics. The acoustic quantities being several orders smaller than their aerodynamic counterparts, the
697 hybrid method should be able to propagate acoustic waves accurately over long distances with very low
698 numerical noise. The aim of this test case is to study the suitability of the hybrid solver to perform acoustic
699 computations and to characterise its dissipation and dispersion errors.

700 In a fluid medium at rest (characterised by $\rho_0 = 1.1765 \text{ kg.m}^{-3}$ and $p_0 = 101320 \text{ Pa}$), a harmonic

701 acoustic source is modeled by a source term \mathbf{S} which is added to the right hand side of Equation (1):

$$702 \quad \mathbf{S}(x, y, t) = A \sin(2\pi f_s t) e^{-\alpha(x^2+y^2)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ c_0^2 \end{pmatrix} \quad (34)$$

703 The shape of the Gaussian source is controlled by $\alpha = \ln 2/4$. In the following, the frequency is fixed to
 704 $f_s = 72$ Hz leading to a wavelength of $\lambda = c_0/f_s = 4.77$ m. The source can then be seen as being acoustically
 705 compact since $\alpha/\lambda \ll 1$. Moreover, low and high acoustic levels are investigated with a sound pressure level
 706 (SPL) of 84 and 144 dB at the edge of the source corresponding to an amplitude of $A = 1 \times 10^{-3}$ and $A = 1$,
 707 respectively. As the source term corresponds to linear acoustics, the two sound levels are chosen in order to
 consider pressure fluctuations which differ by several orders of magnitude.

708 The computational domain, represented on Figure 14, consists in box of size $[42\lambda, 42\lambda, 10\Delta x]$ split into
 709 9 sub-domains of size $[14\lambda, 14\lambda, 10\Delta x]$ each. Again, periodic boundary conditions are implemented in all
 710 directions. The acoustic source, centered at the origin of the domain (i.e. at $(x_s, y_s) = (0, 0)$), will be
 711 generated and computed by the Navier-Stokes finite-volume solver. Owing to its advantageous acoustic
 712 capabilities, the lattice Boltzmann method [7] will be employed in the farfield.
 713 It can be mentioned that the role of each solver can be freely chosen and that different configurations of
 714 the NS and LBM zones have been tested leading to the same level of accuracy on a fine mesh. However,
 715 the hybrid lattice Boltzmann - Navier-Stokes method was found to be the most efficient for the domain
 716 decomposition shown in Figure 14. Therefore, only this configuration will be discussed hereafter.

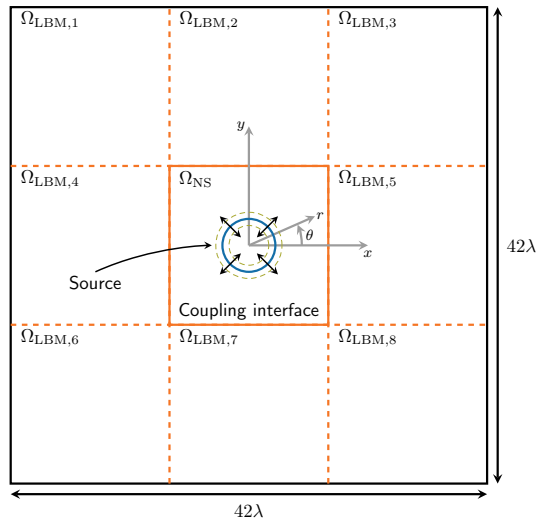


Figure 14: Schematic representation of the computational domain for the harmonic acoustic source radiation test case.

717 When running aeroacoustic computations, one fundamental parameter is the number of mesh points
718 per wavelength $N_{\text{ppw}} = \lambda/\Delta x$. The second order hybrid centered/decentered scheme used to discretise the
719 Eulerian fluxes in the FV solver requires about 15 to 20 points per wavelength in order to correctly propagate
720 acoustic waves. Therefore, in the following, a uniform mesh size will be taken so as to ensure $N_{\text{ppw}} = 18$
721 throughout the entire computational domain. For both solvers, the time step will be set according to the
722 acoustic scaling of Equation (9). Temporal integration will be carried out in the NS-FV solver by means of
723 the 3rd-order explicit Runge-Kutta scheme.

724 Figure 15, which shows the fluctuating pressure field (defined as $p' = p - p_0$) at three successive times,
725 allows to visualise the propagation of the acoustic wave across the coupling interfaces. The figure is split in
726 two parts where the analytical fluctuating pressure field is shown on the upper half and the one computed
727 by the hybrid method is shown on the lower half. Iso-contours of $p' = 0$ have been superimposed so as to
728 highlight the wave fronts. The computation is performed till $t = 18T$, where $T = 1/f_s$, to avoid acoustic
729 interferences caused by the periodic boundary conditions.

730 From Figure 15, it can be deduced that the hybrid lattice Boltzmann - Navier-Stokes method provides
731 good results. Indeed, the circular shape of the iso-contours, which is typical of the directivity of monopolar
732 sources [54], indicates the isotropy of the scheme. In addition, no reflected waves have been generated at the
733 coupling interface between the Navier-Stokes and lattice Boltzmann solver. This confirms that a 3rd-order
734 time interpolation is sufficient for the computation of acoustic test cases.

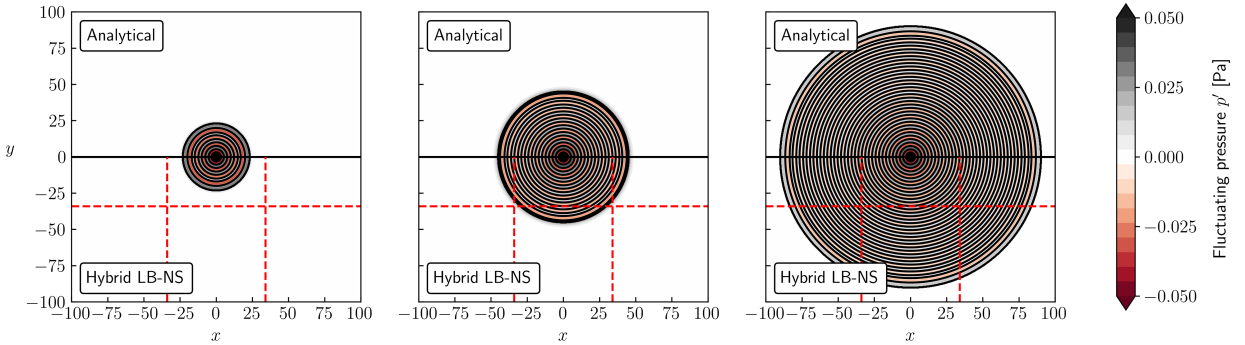


Figure 15: Temporal evolution of the fluctuating pressure field at $t = 4T$, $t = 9T$, and $t = 18T$ (from left to right). In the present case, the amplitude of the source is $A = 1 \times 10^{-3}$.

735 A first estimation of the numerical dispersion induced by the hybrid method can be obtained by tracking
736 the position of the wave fronts along time. This way, one can obtain the numerical speed of sound which
737 appears to be $c_{0,\text{num}} = 347.3 \text{ m.s}^{-1}$. As this value is equal to its theoretical value $c_0 = \sqrt{\gamma RT_0} = 347.3$
738 m.s^{-1} , the dispersion effect on the speed of sound can be considered as negligible.

739 The analysis is now carried out by looking at the fluctuating pressure profile along the $y = 0$ and $x \geq 0$

740 line at the final time of the simulation (see Figure 16). The numerical result is also being compared to the
 741 analytical solution which is known for this case. Indeed, the pressure fluctuations at any point (x, y) in the
 742 far field are given by:

$$p'(x, y, t) = c_0^2 \rho'(x, y, t), \quad \text{where} \quad \rho'(x, y, t) = \frac{A}{\sqrt{kr}} \sin(\omega t - kr). \quad (35)$$

743 In equation (35), k is the wave number defined by $k = \omega/c_0 = 2\pi f_s/c_0$ and $r = |\mathbf{x} - \mathbf{x}_s|$ represents the
 744 distance of any point $\mathbf{x} = (x, y)$ to the source.

745 The good agreement between the numerical result of the hybrid method and the analytical solution can
 746 be observed in Figure 16 in both amplitude and frequency. Moreover, no discontinuity nor oscillations are
 747 observed in the vicinity of the interface thereby validating the information transfer for acoustic phenomena.

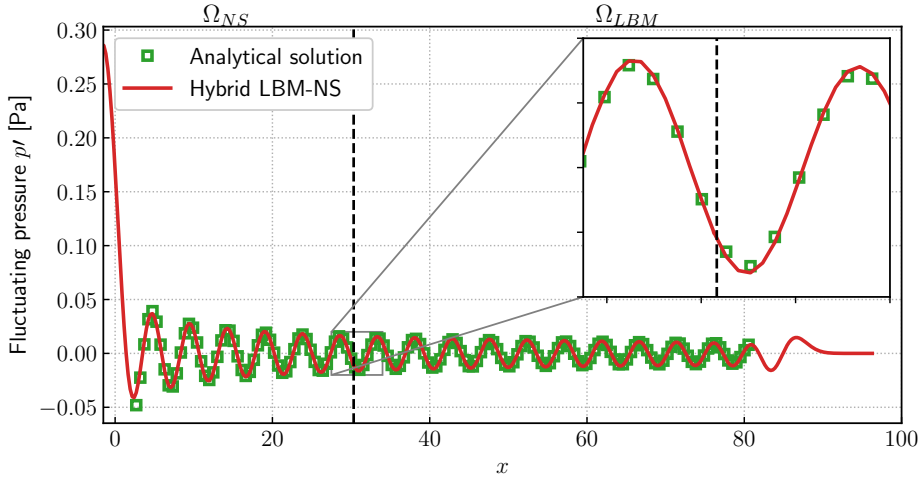


Figure 16: Fluctuating pressure profile along the $y = 0$ and $x \geq 0$ line at the final time of the simulation. The coupling interface is represented by the dashed line. In the present case, the amplitude of the source is $A = 1 \times 10^{-3}$.

748 The dissipation of acoustic waves is linked to the spatial amplitude decay of the waves. For a monopolar
 749 source in radiating in a 2D domain (the flow being homogeneous in the z direction), the decay theoretically
 750 follows a $1/\sqrt{r}$ law. In order to characterise the numerical dissipation of the method, the pressure fluctuations
 751 amplitude peaks along the $y = 0$ and $x \geq 0$ line are reported in a log-log plot as a function of \sqrt{r} in Figure
 752 17. Both sound pressure levels are considered and a linear fitting indicates that the points are all aligned
 753 following a slope of -1 . Consequently, the classical $1/\sqrt{r}$ cylindrical decay is recovered for moderate to loud
 754 acoustic radiation. Moreover, by comparing the wave amplitude with its theoretical value, the numerical
 755 dissipation is negligible in comparison with the viscous molecular damping.

756 The propagation of acoustic waves has been validated on a computational domain relying on a uniform
 757 mesh of 18 points per wavelength. However, as indicated in [7], the lattice Boltzmann method requires less
 758 points per wavelength compared to Navier-Stokes methods to correctly propagate acoustic modes. In light

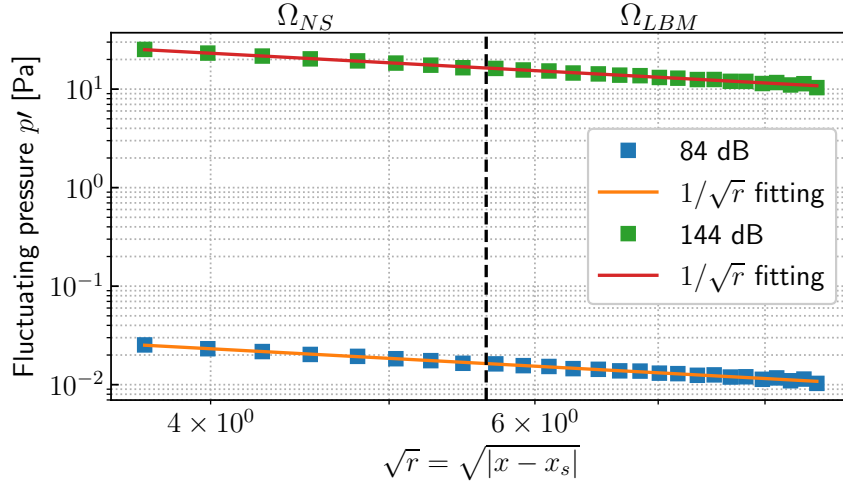
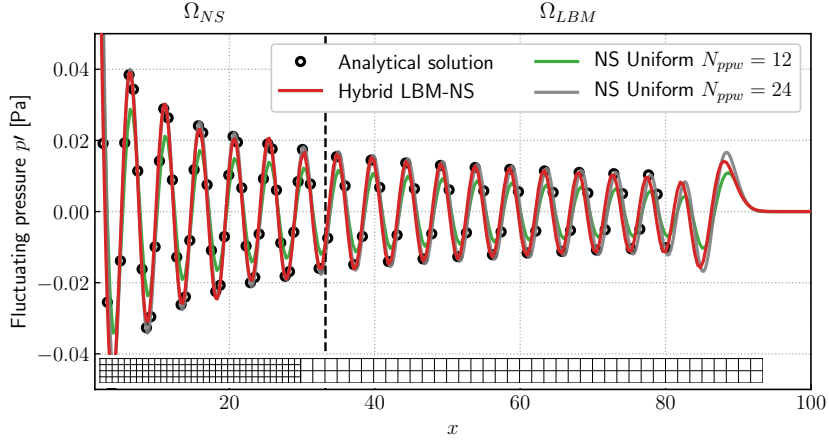


Figure 17: Radial decay of pressure waves for two different sound pressure levels.

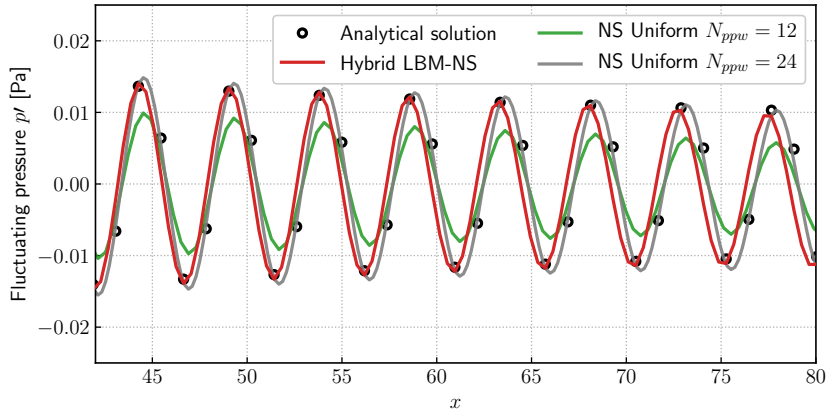
759 of this conclusion, one might be interested in taking advantage of the hybrid lattice Boltzmann - Navier-
760 Stokes method to reduce the cost of acoustic computation. Therefore, the exact same test case will now
761 be computed on a grid where the NS sub-domain and LB sub-domains have a resolution of $N_{ppw} = 24$ and
762 $N_{ppw} = 12$, respectively. The resulting computational domain contains 3 times less points with respect to
763 its uniform counterpart. The aim of this short study is to verify the ability of the coupled solver to properly
764 resolve acoustics while lowering the computational cost.

765 Figure 18 shows the result of 3 computations: a well-resolved full NS computation (with 24 mesh points
766 per wavelength), an under-resolved full NS computation (with 12 mesh points per wavelength), and the
767 hybrid lattice Boltzmann - Navier-Stokes computation on the hybrid mesh priorly introduced. In addition,
768 results will also be compared to the analytical solution of Equation (35). Figure 18a represents the fluctuating
769 pressure profile along the $y = 0$ and $x \geq 0$ line at the final time of the simulation. At the bottom of Figure
770 18a, an overview of the mesh has been added. In the present case, since only 1-to-1 coupling interfaces have
771 been implemented, the change of resolution is performed in the NS domain just before the interface.

772 As expected, the under-resolved NS simulation over-damps the acoustic waves w.r.t. the analytical
773 solution due to the insufficient resolution. By doubling the number of points per wavelength in each direction,
774 the full NS computation retrieves the analytical amplitude decay. However, the most interesting result is
775 provided by the hybrid computation which has 3 times less mesh points than the ones needed for the resolved
776 NS computation. Thanks to the low dissipative property of the LBM, keeping only 12 points per wavelength
777 after the coupling interface is sufficient to properly propagate the sound waves. Figure 18b provides a more
778 detailed view of the different solutions in the far-field, i.e. for $x > 10\lambda$, where it can be seen that the hybrid
779 method leads to a huge improvement in terms of dissipation. However, there is a slight dispersive effect, as is
780 expected for lattice Boltzmann methods [7]. The latter can be quantified and leads to a frequency shift of 0.1



(a)



(b)

Figure 18: Fluctuating pressure profile along the $y = 0$ and $x \geq 0$ line at the final time of the simulation in the case of a non-uniform mesh. The coupling interface is represented by the dashed line. The amplitude of the source is $A = 1 \times 10^{-3}$.

781 Hz which is acceptable in light of the improvement in terms of dissipation. To conclude, the hybrid method
 782 is very promising when computing acoustic problems inasmuch as it helps to decrease the number of points
 783 per wavelength without deteriorating the solution. Even though this test case could have been computed
 784 entirely by the LBM, the following test case will show that when sources become more complex, the use of
 785 NS methods in these regions can bring in an improvement with respect to full LBM computations.

786 4.4. Circular cylinder in an uniform viscous flow

787 The last test case proposed in this paper is the study of the sound generated by a circular cylinder in a
 788 uniform flow. Despite the simple geometry of the obstacle, this test case is still demanding with regard to
 789 the quality of the aerodynamic and acoustic results. Besides, the large disparity between the aerodynamic

790 and acoustic characteristic length scales makes the direct numerical simulation of both the aerodynamics
 791 and acoustics a challenging task in terms of meshing and computational cost.

792 In the following, the proposed hybrid lattice Boltzmann - Navier-Stokes method is compared to full
 793 NS-FV and LBM-HRR computations. While the former are performed using ONERA's CFD suite, the full
 794 LBM-HRR simulation is based on the ProLB commercial solver [65]. Indeed, ONERA's lattice Boltzmann
 795 module is unable to handle multi-resolution computations in its current version preventing simulations from
 796 being carried out at an industrial level. The choice of the ProLB software is thereby motivated by the fact
 797 that it relies on the same core components as the ones presented in Section 2.2.

798 The aim is to rigorously compare the CPU efficiency of each approach when performing the direct
 799 numerical simulation of both the aerodynamics and acoustics simultaneously. From an engineering point of
 800 view, the case was set up for each solver such as to provide an error of 5%, at most, in the estimation of the
 801 aerodynamic forces and the sound pressure level (SPL) at a distance of $150D$ of the source.

802 The flow configuration of Inoue and Hatakeyama [66] is considered. A cylinder of diameter $D = 1$ m
 803 is fixed in a uniform flow. The upstream Mach number M_∞ is set to $M_\infty = 0.2$ and a Reynolds number
 804 $Re = U_\infty D / \nu_\infty$ of 150 is chosen in order to remain below the onset of three-dimensional fluctuations
 805 and turbulent behavior [67]. Since the Mach number is relatively low, temperature dependence of the the
 806 molecular viscosity is not likely to have a significant effect and therefore the latter is taken at a constant
 807 value. The flow configuration is shown in Figure 19.

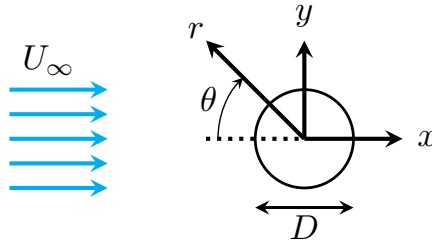
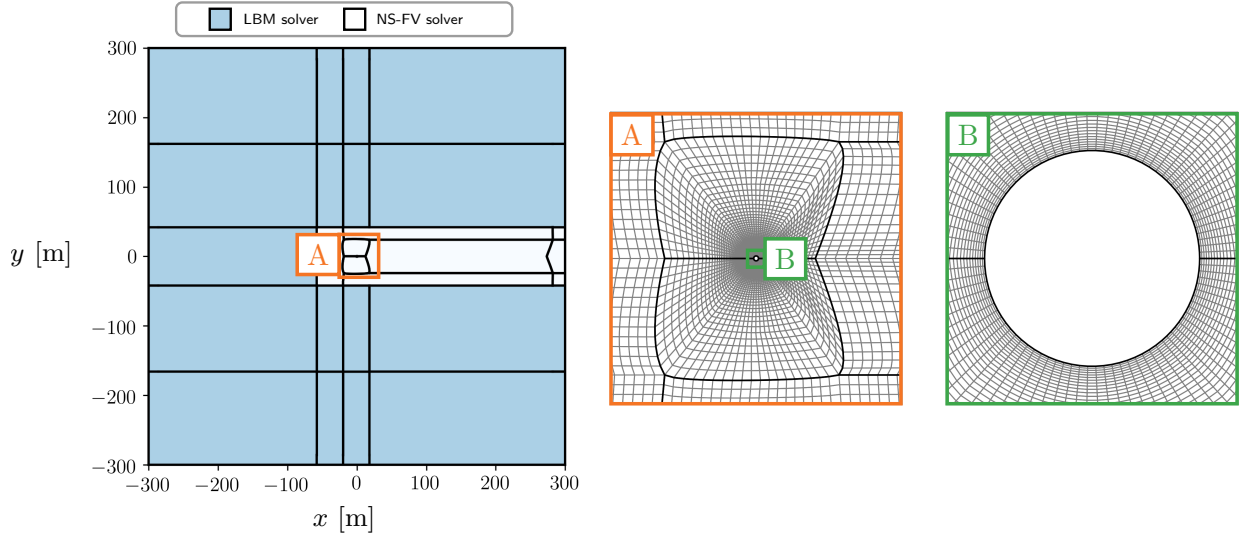


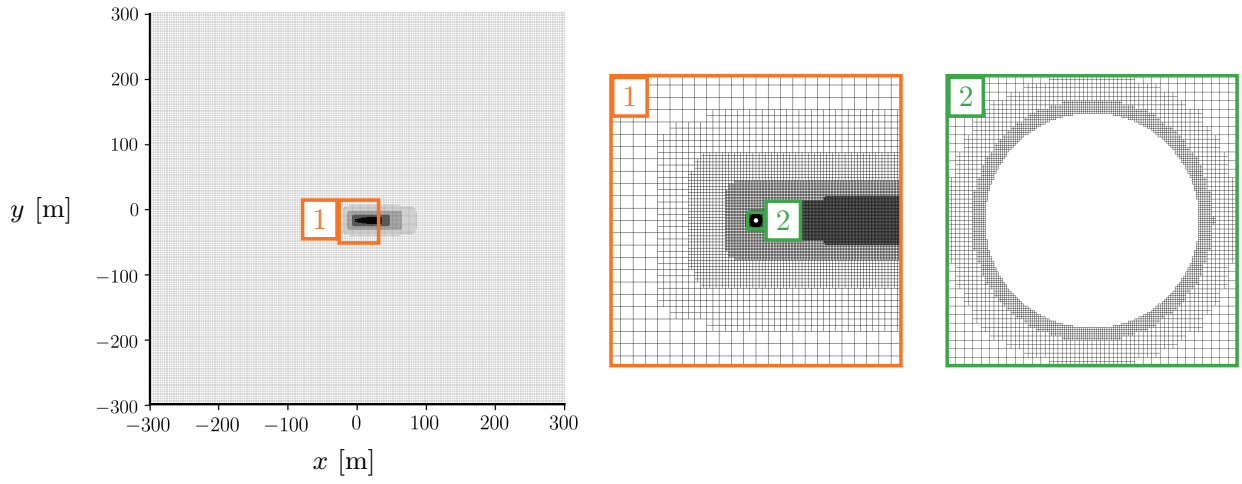
Figure 19: Flow configuration and notations for the study of the flow over a circular cylinder.

808 The computational domain has a size of $[600D, 600D, 10\Delta x]$ and the cylinder is centered at its origin. As
 809 seen in Figure 20, two types of grids are currently used. The first one, shown in Figure 20a, relies on a hybrid
 810 mesh consisting of curvilinear and Cartesian cubic blocks. It is used for both the hybrid lattice Boltzmann
 811 - Navier-Stokes and full Navier-Stokes computations. The second one is a Cartesian cubic grid, which was
 812 automatically generated by ProLB's octree mesher, is used for the full lattice Boltzmann computation and
 813 is shown in Figure 20b. Adiabatic no-slip boundary conditions are used for the cylinder surface and periodic
 814 boundary conditions are applied in the z direction. In addition, non-reflecting far-field boundary conditions
 815 are imposed 300 diameters away from the cylinder in the x and y directions. These boundary conditions

816 are applied in different ways depending on the computations. For the hybrid and full-NS simulations, they
 817 rely on the formalism introduced by Thompson [68] and are applied in NS-FV zones (due to their very
 818 small spatial extent, these are not visible on Figure 20a). For the full LBM simulation, an outlet pressure
 819 is specified at the borders of the computational domain and sponge zones are used in order to prevent
 820 disturbances from reaching the domain boundaries.



(a) Hybrid lattice Boltzmann - Navier-Stokes and full NS-FV computational grid.



(b) Full LB-HRR computational grid.

Figure 20: Computational domains and associated grids for the aeolian tone radiated by the flow past a circular cylinder. (a) Zonal decomposition for the hybrid lattice Boltzmann - Navier-Stokes method. A total of 48 blocks make up the domain. Moreover, closeups of the hybrid mesh are provided. (b) Cartesian cubic grid used for the full LBM-HRR computation. The mesh was automatically generated by ProLB's octree mesher.

821 Concerning the grid spacing, special care is taken on the meshes in Figure 20 to ensure that the error

822 target outlined above is met. In this respect, the first points off the solid surface are placed so as to
 823 remain in the boundary layer. Its thickness is estimated by $\delta \approx 1/\sqrt{Re}$ leading to $\delta \approx 0.08$ for $Re = 150$.
 824 Consequently, on the cylinder surface, the normal grid size is taken to be $\Delta_n = \delta/10$. While for the mesh
 825 in Figure 20a the tangent one can be chosen freely (it is set to $\Delta_s = D/60$ here), the LB mesh forces it
 826 to be equal to Δ_n . Conversely, in the acoustic far-field, the mesh size has to be fixed with respect to the
 827 wavelength associated to the acoustic radiation λ_{ac} . It can be shown that $\lambda_{ac} = D/(St \cdot M_\infty)$ where St
 828 is the Strouhal number associated to vortex shedding leading to $\lambda_{ac} \approx 27D$ in the present case. Owing to
 829 the low dissipative properties of the lattice Boltzmann method, a number of 14 points per wavelength is
 830 chosen (i.e. $\Delta x = \Delta y = \Delta z \approx 2D$) for the full LBM-HRR and hybrid computations. On the other hand,
 831 for the full NS-FV computation, a number of 40 points per wavelength is required to capture the correct
 832 SPL at a distance of $150D$ off the cylinder. In consequence, the computational domain for the hybrid
 833 lattice Boltzmann - Navier-Stokes method is made of about 0.9×10^6 cells, the one for the full LBM-HRR
 834 computation contains 1.6×10^6 Equivalent Fine Cells (EFC) while the the full NS-FV one features 8.5×10^6
 835 cells. Both the NS-FV and the hybrid solver use the same time step in the whole computational domain,
 836 however, this is not the case for LB computations. Indeed, in the standard LBM the time step depends
 837 on the level of refinement, therefore the number of EFCs reflects the workload equivalent of a mesh based
 838 only on the minimal mesh size Δx_{min} . Note that a grid convergence study has been performed on the three
 839 meshes indicating that these are of minimal size.

840 As stated earlier, the zonal decomposition for the hybrid lattice Boltzmann - Navier-Stokes computation
 841 relies on the distinction between aerodynamic and acoustic regions. Thereby, the Navier-Stokes solver is
 842 applied on the body-fitted curvilinear blocks in the vicinity of the cylinder while the lattice Boltzmann
 843 solver is devoted to the computation of the far-field acoustics on the Cartesian blocks (see Figure 20a).
 844 The NS domain extends throughout the wake zone inasmuch as entropy production cannot be considered as
 845 negligible in this flow region and thus the thermodynamic closure of Section 3.2 is no longer valid. Moreover,
 846 due to the large disparity in grid sizes, an implicit time-stepping is used by the Navier-Stokes solver to have
 847 a physical CFL number of $1/\sqrt{3}$ at both sides of the coupling interface. Therefore, to have a fair comparison,
 848 the full Navier-Stokes computation will also be performed by means of an implicit time-stepping scheme.

849 The use of the hybrid lattice Boltzmann - Navier-Stokes method for such a computation may be very
 850 beneficial. Indeed, Figure 20 highlights the fact that the geometry of the cylinder is much more accurately
 851 represented with a body-fitted mesh in comparison with the Cartesian “staircase” mesh imposed by standard
 852 lattice Boltzmann methods. Even though some authors have proposed solutions to overcome the limit of
 853 Cartesian grids for LBMs [69, 70], the exact advection property of the LBM is lost, thereby adding numerical
 854 dissipation, and no aeroacoustic studies of such approaches have yet to be published. In addition, one more
 855 advantage of the hybrid method can be highlighted. Indeed, mesh-refinement in the lattice Boltzmann

856 framework may produce spurious noise [62, 71] due to the abrupt non-conformal change of resolution by a
 857 factor 2. In the present case, the flexibility of meshing offered by the use of a Navier-Stokes solver enables
 858 the use of grid stretching when moving away from the cylinder. Thus, a smooth transition from the near-wall
 859 very fine resolution to the acoustic grid size can be achieved.

860 As in the case of the harmonic acoustic source, one can switch the role of the solvers and still obtain the
 861 same quality of solution since the hybrid method is developed so as to provide a two-way coupling. However,
 862 in the light of the brief discussion outlined above, switching the role of the solvers might not be of great
 863 practical interest since each solver would be in its most unfavourable case.

864 4.4.1. Aerodynamic study

865 First, the aerodynamic results of the hybrid method are analysed. At the initial stages of the time
 866 evolution, an x -axis symmetric wake composed of two counter-rotating vortices develops downstream of
 867 the cylinder. The base flow destabilizes and transitions to an asymmetric von-Karman vortex-street. The
 868 alternating vortex shedding from the upper and lower sides of the cylinder is shown in Figure 21 where
 869 the instantaneous vorticity field is shown at two distinct instants of time. Consequently, negative and
 870 positive pressure pulses are produced alternately from the upper and lower sides of the cylinder, resulting
 871 in fluctuating aerodynamic forces.

872 The parameters of interest are the lift and drag coefficients (denoted by C_l and C_d respectively). These
 873 coefficients are defined by:

$$C_l = \frac{F_L}{\frac{1}{2}\rho_\infty u_\infty^2 D} \quad \text{and} \quad C_d = \frac{F_D}{\frac{1}{2}\rho_\infty u_\infty^2 D}. \quad (36)$$

874 In Equation (36), F_L is the lift force and F_D the drag force. The uniform flow being aligned with the x -axis,
 875 the drag corresponds to the horizontal component and the lift force to the vertical component of the total
 876 aerodynamic force acting on the cylinder. This force is computed as the sum of two contributions: a locally
 877 normal force due to pressure and a friction force due to viscous effects and the no-slip condition. For the sake
 878 of completeness, and to assess the quality of the near-wall resolution, the time averaged pressure coefficient
 879 C_p is also investigated. It is defined as :

$$C_p = \frac{p_{\text{cyl}} - p_0}{\frac{1}{2}\rho_\infty u_\infty^2}, \quad (37)$$

880 where p_{cyl} is the local value of pressure on the surface of the cylinder.

881 Figure 22a, shows the time variations of C_l and C_d for the hybrid, full NS, and full LBM-HRR compu-
 882 tations. As stated earlier, the flow around the cylinder is characterised by strong oscillating aerodynamic
 883 efforts. The amplitude of the lift coefficient C_l' is much larger than the one of the drag coefficient C_d' sug-
 884 gesting that the associated sound will mainly be due to the lift dipole [66]. The mean value of C_d , denoted
 885 by $\overline{C_d}$ hereafter, is 1.378 for both the hybrid and NS computations and 1.41 for the LB simulation. All these
 886 values are in agreement with the one obtained by Inoue *et al.* [66], i.e $\overline{C_d} = 1.3805$.

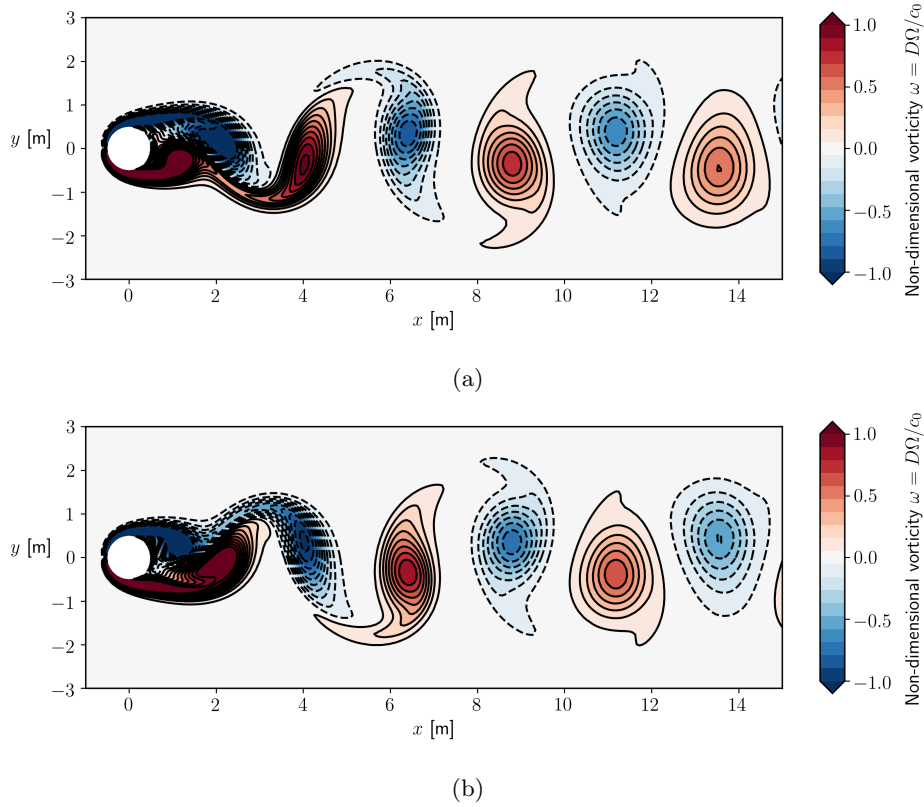


Figure 21: Time evolution of the non-dimensional vorticity field $\omega = D\Omega/c_0$ where $\Omega = 1/2(\partial_x u_y - \partial_y u_x)$ computed by the hybrid lattice Boltzmann - Navier-Stokes method. (a) Vorticity at the time of minimum C_l and (b) vorticity at the time of maximum C_l . The contour levels are from $\omega_{min} = -1.0$ to $\omega_{max} = 1.0$ with an increment of 0.1. — : $\omega > 0$ and - - - : $\omega < 0$.

887 The periodic behaviour of both the vortex shedding and the oscillation of the lift coefficient can be
 888 described through their characteristic frequency f . Most of the time, this frequency is expressed by means
 889 of the non-dimensional Strouhal number $St = fD/U_\infty$. According to Figure 22a, the Strouhal number in
 890 the case of the hybrid lattice Boltzmann - Navier-Stokes computation is 0.1817. One can also notice that the
 891 drag coefficient C_d is oscillating at twice the frequency of C_l . All the aerodynamic results relative to the lift
 892 and drag coefficients are summarised in Table 5 and show good agreement with two reference computations
 893 with a Navier-Stokes [66] and lattice-Boltzmann solver [72] respectively. Moreover, as expected, the hybrid
 894 method and the full NS computations lead to the same aerodynamic results. In addition, Figure 22b shows
 895 the time-averaged pressure coefficient C_p along the cylinder top surface and, regardless of the numerical
 896 method, it is in good agreement with the reference DNS of Inoue *et al.* [66].

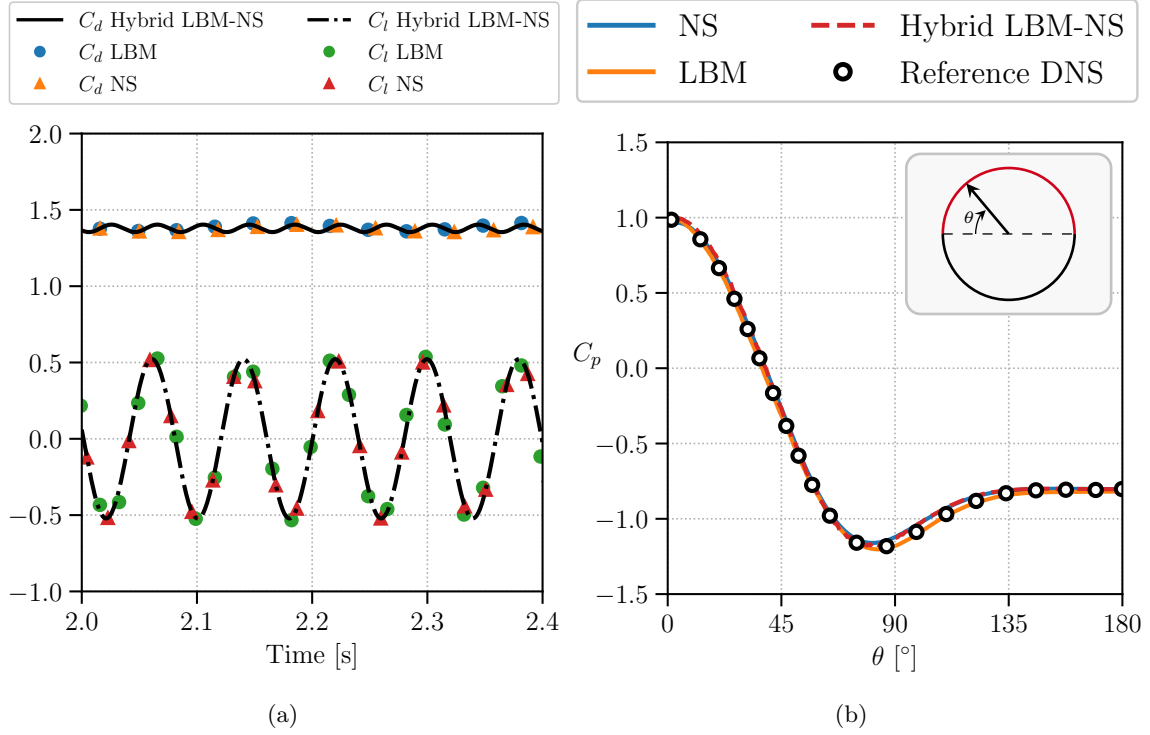


Figure 22: Force coefficients for the flow around a circular cylinder at $M = 0.2$ and $Re = 150$. (a) Time evolution of the lift C_l and drag C_d coefficients. (b) Time-averaged pressure coefficient C_p compared with the reference DNS of Inoue *et al.* [66].

	$\overline{C_d}$	C'_l	C'_d	St
Hybrid LBM-NS	1.378	0.522	0.0248	0.1817
Full NS	1.378	0.522	0.0248	0.1817
Full LBM (ProLB)	1.41	0.537	0.027	0.184
Inoue <i>et al.</i> [66]	1.3805	0.52	0.026	0.183
Lafitte <i>et al.</i> [72]	1.39	0.56	0.028	0.185

Table 5: Comparison of the characteristic parameters with the Navier-Stokes DNS computation of Inoue *et al.* [66] and a lattice Boltzmann computation by Lafitte *et al.* [72].

897 4.4.2. Aeroacoustic study

898 The aerodynamic near-field being validated, the flow induced noise is now analysed. Sound pressure
899 waves are indeed generated by the fluctuating lift force in response to the vortex shedding mechanism. As
900 a consequence, a strong tonal noise at the exact same frequency is expected. In the following, acoustic
901 radiation will be described through the fluctuating pressure field defined (following [66]) as $\Delta\tilde{p}(x, y, t) =$
902 $\Delta p(x, y, t) - \Delta p_{\text{mean}}(x, y)$. Here $\Delta p = p - p_0$, where p is the total pressure and p_0 its reference value. The
903 effect of the time-averaged fluctuating pressure Δp_{mean} has been extensively discussed in [66].

904 Figure 23a shows the instantaneous fluctuating pressure field $\Delta\tilde{p}$ in the whole computational domain for
 905 the hybrid lattice Boltzmann - Navier-Stokes computation. It can be seen that pressure waves with opposite
 906 signs are generated from both upper and lower sides of the cylinder. This indicates the dipolar nature of
 907 the radiated acoustic field. Moreover, the pulses propagate radially with time following a propagation angle
 908 θ close to its theoretical value $\theta_p = \arccos(M_\infty) = \pm 78.5^\circ$.

909 In Figure 23b, iso-contours of the instantaneous fluctuating pressure field $\Delta\tilde{p}$ are represented in the
 910 vicinity of the coupling interface. The same conclusion as for the previous test cases can be made: the
 911 pressure field remains continuous through the interface as no oscillations nor discontinuities in the contour-
 912 lines are exhibited. Inasmuch as implicit time-stepping is used by the NS-FV solver, care has been taken
 913 to perform enough sub-iterations to fully damp the spurious effect of the explicit boundary treatment. In
 914 addition, both plots are symmetrical with respect to the $y = 0$ line and of opposite sign.

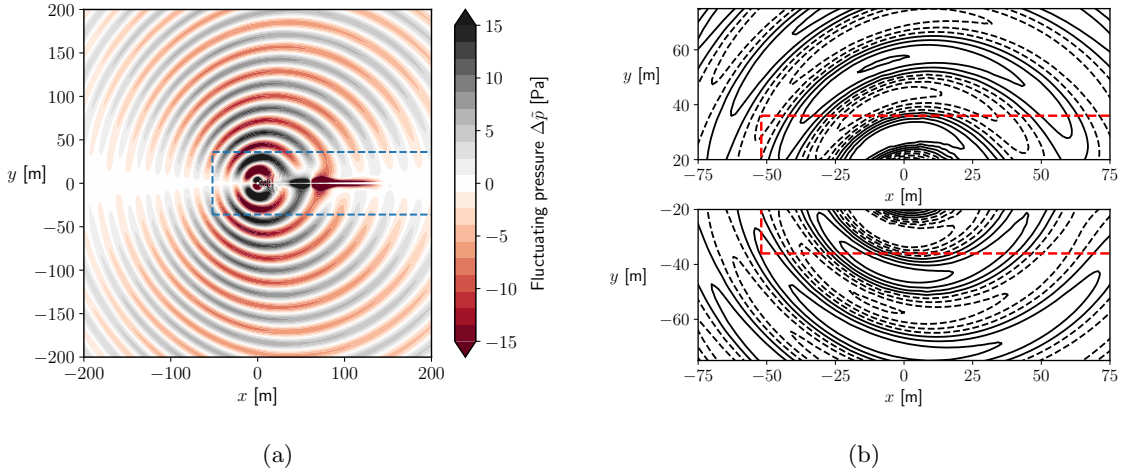


Figure 23: Visualisation of the instantaneous fluctuating pressure field $\Delta\tilde{p}$. (a) Pressure fluctuations in the computational domain between -15 Pa and 15 Pa. The interface between both solvers is indicated by the dashed lines. (b) Iso-contours of pressure fluctuations. The contour levels are from $\Delta\tilde{p}_{min} = -15$ Pa to $\Delta\tilde{p}_{max} = 15$ Pa with an increment of 3.75 Pa. — : $\Delta\tilde{p} > 0$ and - - - : $\Delta\tilde{p} < 0$.

915 Inoue *et al.* [66] have shown that the sound radiated by the cylinder is dominated by the lift dipole.
 916 The results in Figure 23 tend to verify this conclusion. However, in order to validate more quantitatively the
 917 quality of the acoustic field computed with the hybrid lattice Boltzmann - Navier-Stokes method, the polar
 918 plot of the root mean square of the fluctuating pressure is studied. The latter quantity is defined as:

$$\Delta p_{\text{rms}}(x, y) = \sqrt{[\overline{\Delta p^2}](x, y) - [\overline{\Delta p}(x, y)]^2}, \quad (38)$$

919 where $[\overline{\Delta p^2}]$ is the time average of the squared fluctuating pressure and $[\overline{\Delta p}(x, y)]$ the time average of the
 920 fluctuating pressure. Both values were computed for the last 10 periods of the computation when all the

921 aerodynamic coefficients reached the statistical steady state.

922 Figure 24 shows the polar diagram of the root mean square pressure fluctuation Δp_{rms} at a distance of
 923 $r = 75D$ and $r = 150D$ of the cylinder for the hybrid, full NS, and full LBM-HRR computations. Note
 924 that pressure fluctuations have been made non-dimensional to compare the present results with literature.
 925 Indeed, Inoue *et al.* [66] provide a polar plot at $r = 75D$ and, the one at $r = 150D$ is easily obtained by a
 926 scaling factor of $\sqrt{75/150}$ to take into account the $r^{-1/2}$ dependence of the amplitude of the sound waves.
 927 In each diagram, the radial length from the origin represents the magnitude on a linear scale where the
 928 outermost circle corresponds to a value of $\frac{\Delta p_{\text{rms}}}{\rho_0 c_0^2} = 1 \times 10^{-4}$ and $\frac{\Delta p_{\text{rms}}}{\rho_0 c_0^2} = 0.75 \times 10^{-4}$ respectively.

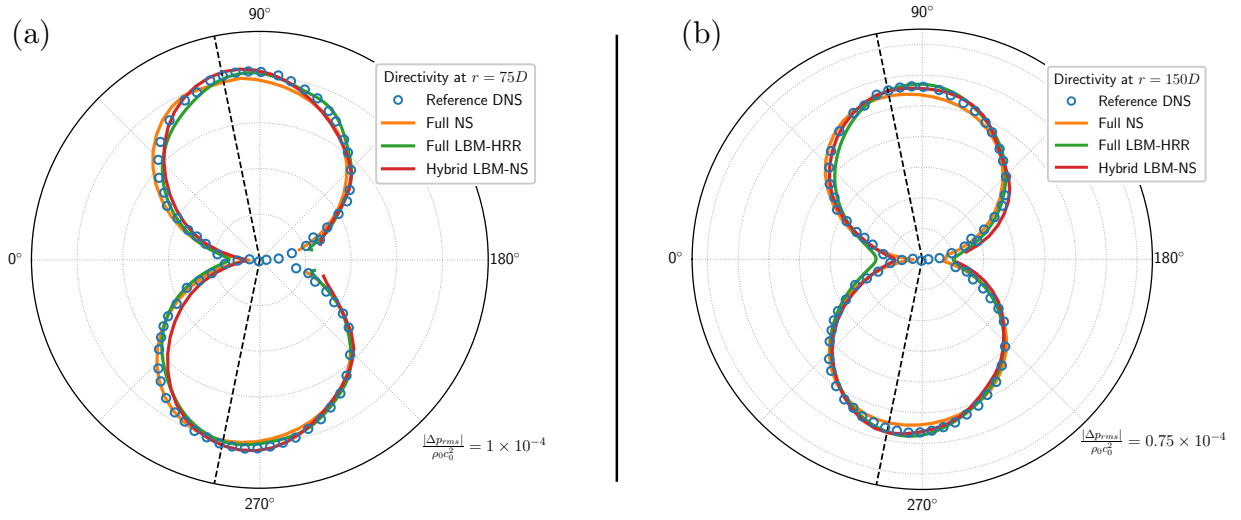


Figure 24: Polar plots of the root mean square of the non-dimensional fluctuating pressure at a distance of (a) $r = 75D$, and (b) $r = 150D$ of the cylinder. The symbols (○) denote the results of the reference computation of Inoue *et al.* [66].

929 Both plots in Figure 24 confirm the dipolar nature of the sound field. The directivity of the sound
 930 waves agrees with its theoretical value of $\theta_p = \pm 78.5^\circ$ due to the Doppler effect as shown by the dashed
 931 lines (---). All three computations are superimposed to the directivity obtained by the reference DNS
 932 [66] thereby further validating the meshes in Figure 20. It is noteworthy that, owing to the low numerical
 933 dissipation of the lattice Boltzmann method, the pressure fluctuations can be propagated over long distances
 934 with a very limited number of points per wavelength (14 in the present case). Therefore, the hybrid lattice
 935 Boltzmann - Navier-Stokes method is very promising when far-field aeroacoustics have to be finely captured.

936 The last acoustic feature which is studied is the spatial decay of the sound pressure waves. As already
 937 stated with the case of the harmonic source of Section 4.3, the pressure peaks tend to decay in proportion
 938 to $r^{-1/2}$ with increasing r in two dimensions of space. In order to assess this property in the present case,
 939 instantaneous snapshots of the fluctuating pressure $\Delta \tilde{p}$ at four successive instants are plotted against the
 940 distance r to the center of the cylinder at $\theta = 90^\circ$ (see Figure 25a). As seen from Figure 25a, pressure waves

941 propagate radially with time. In addition, the interface between both solvers being shown by the vertical
 942 dashed line, it can be confirmed that no reflected waves appear in the vicinity of the coupling interface. To
 943 get more insight into the pressure fluctuations decay, the negative and positive peak values of $\Delta\tilde{p}$ are plotted
 944 against r in a log-log scale on Figure 25b. A reference line proportional to $r^{-1/2}$ has also been added to this
 945 figure. As readily seen from Figure 25b, the pressure peaks computed with the hybrid lattice Boltzmann -
 946 Navier-Stokes method tend to decay following a $r^{-1/2}$ slope with increasing r .

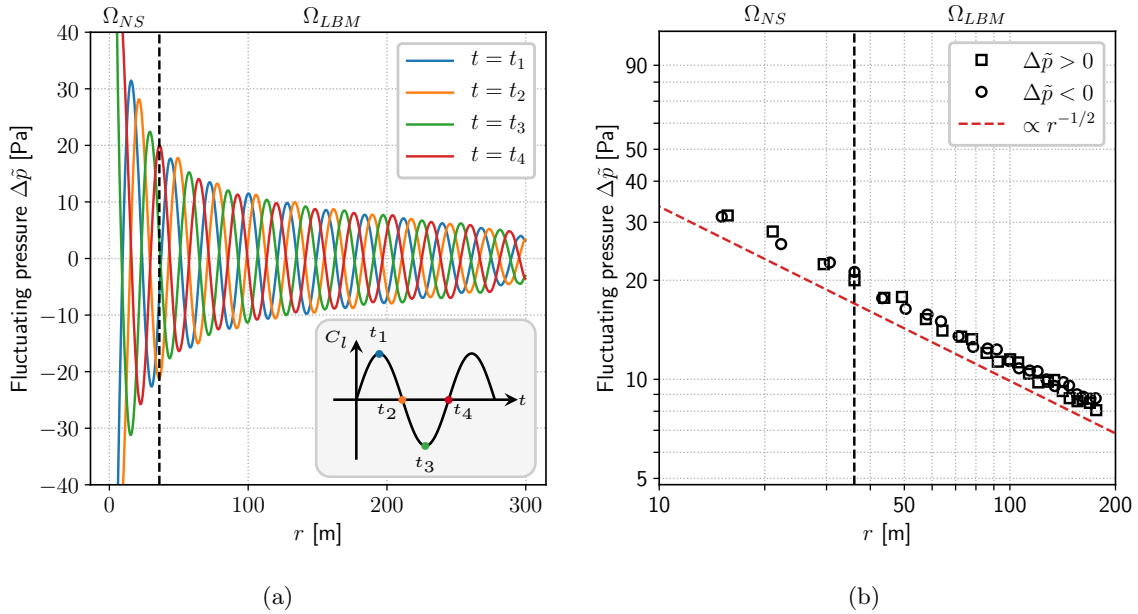


Figure 25: Propagation and decay of sound waves at $\theta = 90^\circ$. (a) Radial propagation of sound waves with time. 4 different instants corresponding to specific values of the lift coefficient C_l are chosen to fully describe one period evolution. (b) Decay of both the negative and positive pressure peaks. Comparison with the theoretical decay $\propto r^{-1/2}$.

947 4.4.3. Computational cost

948 Some of the advantages of the hybrid lattice Boltzmann - Navier-Stokes method on this specific test case
 949 have already been mentioned such as the flexibility on meshing, time-stepping, and the overall accuracy of
 950 the approach. The computational cost of the hybrid solver is now investigated. The aim here is not to carry
 951 out an in-depth study of the solver's HPC capabilities (in terms of scaling, parallelism, etc.), but rather
 952 to give an order of magnitude of the performance, on this specific test case, of both the Navier-Stokes and
 953 lattice Boltzmann solvers and to draw some conclusions about the benefits of the hybrid lattice Boltzmann
 954 - Navier-Stokes method. All the computations were run on ONERA's development cluster on one bi-socket
 955 node consisting of two 12-core Intel Broadwell processors (E5-2650v4, 2.2 GHz) with 128 GB of memory.

956 First of all, the effective performance of each individual solver of ONERA's CFD suite is studied. While
 957 the in-house lattice Boltzmann solver with HRR collision operator achieves 12.6 MCUPS (Million Cells

958 Update Per Second) per core, the Navier-Stokes solver scales at 3.4 MCUPS per core in the explicit case
 959 and at 0.3 MCUPS per core in the implicit case. In comparison, the cost of the hybrid lattice Boltzmann -
 960 Navier-Stokes method is not constant since it directly depends on the ratio between NS and LB cells as well
 961 as the time-stepping scheme used by the NS solver. For instance, on the mesh in Figure 20, where 25% of
 962 the cells are NS ones, the hybrid method reaches 8.9 MCUPS per core when coupled with the NS-implicit
 963 solver. These numbers are close to the theoretical value of the performance of the hybrid method:

$$S_{\text{hyb}} = \frac{N_{\text{NS}}}{N_{\text{tot}}} S_{\text{NS}} + \frac{N_{\text{LBM}}}{N_{\text{tot}}} S_{\text{LBM}} \quad (39)$$

964 where S_{hyb} , S_{NS} , and S_{LBM} are the computational speeds in MCUPS and where $N_{\text{NS}}/N_{\text{tot}}$ and $N_{\text{LBM}}/N_{\text{tot}}$
 965 represent the proportion of cells computed by the NS and the LB solver respectively. As a consequence, one
 966 can conclude that the coupling process has a negligible computational footprint.

967 The three approaches are now compared in terms of CPU efficiency. As stated earlier, the metric of
 968 interest is the total CPU time $T_{\text{CPU}}^{5\%}$ required to capture both the sound pressure level (SPL) at a distance
 969 of $r = 150D$ of the cylinder and the correct boundary layer (in terms of C_l , C_d and C_p) within a 5%
 970 error-margin with respect to their theoretical values. The results are summarised in Table 6 where $T_{\text{CPU}}^{5\%}$
 971 is provided for the hybrid LBM-NS, full NS, and full LBM-HRR computations respectively. Note that the
 972 total CPU time only takes into account the effective time of computation and does not include the time
 973 needed for the evaluation of post-processing quantities such as statistics. Moreover, to ease the comparison,
 974 the relative cost of each method with respect to the hybrid approach and the total number of cells in the
 975 computational grid are also reported.

976 It should be noted that the computational time indicated in Table 6 for the full LBM-HRR computation is
 977 an estimated one. As a matter of fact, the ProLB solver which is used for the aerodynamic and aeroacoustic
 978 studies (see Sections 4.4.1 and 4.4.2) is an unstructured one, thus its computational time is not representative
 979 of the performance which can be achieved by using the in-house LB solver. In addition, the comparison
 980 of the CPU time obtained with ProLB with the hybrid method would be unfair since the latter is using
 981 ONERA's in-house structured lattice Boltzmann software. Thereby, the value of $T_{\text{CPU}}^{5\%}$ provided in Table
 982 6 for a full LBM-HRR simulation was computed by dividing the number of EFC by the performance in
 983 MCUPS of ONERA's LB solver to reach converged statistics.

984 From Table 6, one can easily see that the hybrid lattice Boltzmann - Navier-Stokes methods is the most
 985 efficient one to reproduce both the aerodynamics and the acoustics within the targeted error range. Indeed,
 986 due to the high dissipation of acoustic modes by full Navier-Stokes computations, the number of mesh points
 987 per wavelength has to be increased in the acoustic region leading to a 10-fold increase in the number of cells
 988 in the computational grid with respect to the mesh of the hybrid method. Hence, for the same level of
 989 accuracy, the hybrid lattice Boltzmann - Navier-Stokes method is about 2.7 times faster than a full NS-FV

	N_{cells} [M]	Δt [s]	$T_{\text{CPU}}^{5\%}$ [s]	$T_{\text{CPU}}^{5\%} / (T_{\text{CPU}}^{5\%})_{\text{hyb}}$
Hybrid LBM-NS	0.91	3.3×10^{-3}	3877	1
Full LBM-HRR [†]	1.6 (EFC)	1.7×10^{-5}	7893	2.04
Full NS-Implicit	8.5	1.2×10^{-2}	10 442	2.69

Table 6: Comparison of the computational costs of the hybrid lattice Boltzmann method with full NS-FV and full LB computations to capture both the correct SPL at a distance of $r = 150D$ of the source and the correct boundary layer within a 5% error-margin w.r.t their theoretical values. [†]: The total CPU time $T_{\text{CPU}}^{5\%}$ for the LBM computation is an estimated one.

990 implicit computation. This clearly shows the advantage of using the LBM as an acoustic propagator.

991 Conversely, to highlight the benefit of using the NS-FV solver in the near-wall region, the computational
992 cost of the hybrid method is compared to a full LBM-HRR computation. As shown in Table 6, a full
993 LBM-HRR computation is about 2 times slower than the one performed with the hybrid lattice Boltzmann
994 - Navier-Stokes method. This increase in computational time is mainly explained by the fact that wall-
995 resolved computations are expensive within the LB framework since the method is restricted to cubic cells
996 $\Delta x = \Delta y = \Delta z$ and evolving with an explicit time-stepping scheme at a fixed CFL number. To give an
997 order of magnitude, about 95% of the cells in the full LB mesh (1.5×10^6 cells) are located within the
998 cylinder’s near wake region. In contrast, for the hybrid lattice Boltzmann - Navier-Stokes mesh, only 12%
999 of the cells (0.1×10^6 cells) make up this flow region. Moreover, this value of $T_{\text{CPU}}^{5\%}$ is just a lower bound
1000 estimate since it does not take into account any specific algorithmic treatment at the interface between the
1001 resolution domains. This clearly shows the advantage of using the NS-FV solver in the near-wall region
1002 when the levels of accuracy provided by wall functions is insufficient thereby justifying the need of wall
1003 resolved computations.

1004 This short discussion on computational costs helps to underline one more benefit of the hybrid method.
1005 Since each numerical method is the most efficient in different regions of the flow, combining them spatially
1006 and temporally can reduce the CPU time needed to compute multi-physics problems. Note that the hybrid
1007 method could be even more efficient by using explicit local time stepping [73]. Indeed, as most of the time
1008 computational time spend in evolving the NS-FV implicit domains, removing the implicit time stepping
1009 might lead to another increase in terms of CPU efficiency. This is planned for future work.

1010 5. Conclusions

1011 In this paper, a hybrid lattice Boltzmann - Navier-Stokes method for unsteady aerodynamic and aero-
1012 acoustic simulations has been proposed. This method relies on the partitioning of the computational domain
1013 into distinct non-overlapping regions where either the lattice Boltzmann or a compressible finite-volume
1014 Navier-Stokes method is applied. The technical difficulties of a two-way coupling between both solvers have

1015 been discussed and it has been shown that such procedure is not straightforward as each numerical methods
1016 relies on its own set of variables. The heart of the coupling methodology lies on the way the LB distribu-
1017 tion functions are obtained from macroscopic flow quantities and their derivatives. In the present study,
1018 the distribution functions are obtained through a first-order Chapman-Enskog expansion without any prior
1019 approximation on their expressions. One of the original features of this strategy is the direct link which
1020 has been drawn between the reconstruction procedure and the HRR collision operator. Moreover, special
1021 care is taken when coupling various time-stepping schemes with the lattice Boltzmann stream and collide
1022 algorithm. A detailed evaluation of time-interpolation methods is therefore performed in the context of
1023 computational aeroacoustics.

1024 The hybrid lattice Boltzmann - Navier-Stokes method is assessed on four benchmark cases. While the
1025 gaussian 1D acoustic wave shows the influence of each component of the coupling procedure, the convection
1026 of a barotropic vortex demonstrates the feasibility of the proposed approach in an unsteady context and
1027 shows a second order convergence of the hybrid method in both space and time. The acoustic radiation of
1028 a harmonic source and the flow past a circular cylinder help to demonstrate the benefits and capabilities of
1029 the the hybrid method. Firstly, the low dissipation of the lattice Boltzmann method allows to reduce the
1030 number of mesh points per wavelength in acoustic regions and thereby the total size of computational grids.
1031 Secondly, when computing the flow around obstacles, the use of the Navier-Stokes method in near-wall
1032 regions enables to accurately describe complex geometries through the use of body-fitted meshes. Hence,
1033 complex flow phenomena are directly resolved and no wall laws are needed as is commonly the case in the
1034 lattice Boltzmann framework. Thirdly, and most importantly, the overall computational cost is reduced by
1035 the use of the hybrid approach for simulations where both the aerodynamics and acoustics are computed
1036 simultaneously. When considering the aeolian tone of the flow past a circular cylinder, using the coupled
1037 solver helps to decrease the total CPU time by a factor two with respect to full Navier-Stokes and lattice
1038 Boltzmann computations. To the authors knowledge, the present hybrid lattice Boltzmann - Navier-Stokes
1039 method is the first one being introduced and validated in the context of aeroacoustic computations.

1040 Future work will consist in extending the strategy to more complex interface geometries such as mesh
1041 refinements and non-conforming grids. It is also believed that the lattice Boltzmann method could be an
1042 interesting candidate for the propagation of turbulent wakes over long distances. Therefore, temperature
1043 fluctuations and compressibility effects have to be accounted for in the lattice Boltzmann solver or at least
1044 at the interface. In addition, exploring other time stepping strategies might help to improve the overall
1045 computational cost of the hybrid solver. As a perspective, it would then be interesting to evaluate the
1046 hybrid lattice Boltzmann - Navier-Stokes method on representative aeronautical applications, such as cavity
1047 or trailing edge noise, where both the boundary layer and the acoustics have to be accurately characterised.

1048 **Acknowledgements**

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1050 **Appendix A. Mixed Taylor/Chapman-Enskog expansion**

1051 In this appendix, the mixed Taylor/Chapman-Enskog expansion of the lattice Boltzmann scheme is
 1052 detailed. Therefore, Equation (11) is written in a slightly different way by introducing the time-step Δt and
 1053 by recasting the regularised collision operator in a BGK-like fashion:

$$g_i(\mathbf{x} + \boldsymbol{\xi}_i \Delta t, t + \Delta t) = g_i(\mathbf{x}, t) - \frac{\Delta t}{\bar{\tau}} (g_i(\mathbf{x}, t) - g_i^{eq}(\mathbf{x}, t)) + \frac{\Delta t}{2} \psi_i(\mathbf{x}, t). \quad (\text{A.1})$$

1054 Assuming that the time step Δt is small, a second-order Taylor-expansion is performed on the left hand-side
 1055 of equation (A.1) yielding:

$$\Delta t \left[\frac{\partial}{\partial t} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}} \right] g_i + \frac{\Delta t^2}{2} \left[\frac{\partial}{\partial t} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}} \right]^2 g_i = -\frac{\Delta t}{\bar{\tau}} (g_i - g_i^{eq}) + \frac{\Delta t}{2} \psi_i + \mathcal{O}(\Delta t^3), \quad (\text{A.2})$$

1056 where the explicit space and time dependance of the distribution functions and of the corrective term has
 1057 been dropped for the sake of clarity. The second-order derivative term appearing in equation (A.2) has
 1058 often been neglected without any rigorous explanation [15, 18]. In fact, it can be discarded by subtracting
 1059 $(\Delta t/2)(\partial_t + \boldsymbol{\xi}_i \partial_{\mathbf{x}})$ applied to the equation itself. Dividing the final equation by Δt , one obtains:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}} \right] g_i = -\frac{1}{\bar{\tau}} g_i^{neq} + \frac{1}{2} \psi_i + \frac{\Delta t}{2\bar{\tau}} \left[\frac{\partial}{\partial t} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}} \right] g_i^{neq} + \frac{\Delta t}{2} \left[\frac{\partial}{\partial t} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}} \right] \psi_i + \mathcal{O}(\Delta t^2), \quad (\text{A.3})$$

1060 where $g_i^{neq} = g_i - g_i^{eq}$. At this stage, a first relation defining the off-equilibrium distribution functions is
 1061 obtained. Nevertheless, the aim is to express this off-equilibrium term as a function of the macroscopic
 1062 variables. Therefore, a Chapman-Enskog expansion [74] is now performed.

1063 The Chapman-Enskog expansion [74] is a mathematical tool commonly used to understand the link
 1064 between the Boltzmann equation and the macroscopic Navier-Stokes equations. The formalism, which has
 1065 been developed in the continuous case, consists in expanding the space- and time-continuous distribution
 1066 functions f_i in powers of a small parameter ϵ which can be identified to the Knudsen number Kn :

$$f_i[\rho(\mathbf{x}, t), \rho \mathbf{u}(\mathbf{x}, t)] = \sum_{n=0}^{\infty} \epsilon^n f_i^{(n)}[\rho(\mathbf{x}, t), \rho \mathbf{u}(\mathbf{x}, t)]. \quad (\text{A.4})$$

1067 The expansion parameter ϵ indicates that $f_i^{(1)}/f_i^{(0)} = \mathcal{O}(\text{Kn})$, $f_i^{(2)}/f_i^{(0)} = \mathcal{O}(\text{Kn}^2)$ and so on. Moreover,
 1068 as any collision operator must conserve mass and momentum, the following conditions are imposed on the
 1069 high-order terms of the expansion:

$$\sum_{i=1}^q f_i^{(k)} = 0 \quad \text{and} \quad \sum_{i=1}^q \boldsymbol{\xi}_i f_i^{(k)} = \mathbf{0} \quad \text{for } k \geq 1. \quad (\text{A.5})$$

1070 These conditions are also called the solvability conditions. Mathematically, they represent the search for
 1071 normal solutions of the Boltzmann equation.

1072 Besides the expansion of the space- and time-continuous distribution functions f_i , the time and space
 1073 derivative operators are also expanded in terms of ϵ :

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} \quad \text{and} \quad \frac{\partial}{\partial \mathbf{x}} = \epsilon \frac{\partial}{\partial \mathbf{x}_1}. \quad (\text{A.6})$$

1074 The time derivative is expressed as the sum of a fast convective time scale t_1 and a slow diffusive time scale
 1075 t_2 whereas the space derivative is expanded to the first order as only the large scale dynamics are of interest.

1076 The expansion introduced in Equation (A.4) can not be applied as is on Equation (A.3) owing to the
 1077 variable change performed when discretising the DVBE in both space and time. Indeed, in the case of the
 1078 corrected HRR collision operator, the space- and time-discrete distribution functions g_i are defined as:

$$g_i = f_i + \frac{\Delta t}{2\tau} (f_i - f_i^{eq}) - \frac{\Delta t}{2} \psi_i. \quad (\text{A.7})$$

1079 By combining Equation (A.4) and Equation (A.7), the following expansion is obtained:

$$g_i = \sum_{n=0}^{\infty} \epsilon^n f_i^{(n)} + \frac{\Delta t}{2\tau} \left(\sum_{n=0}^{\infty} \epsilon^n f_i^{(n)} - f_i^{eq} \right) - \frac{\Delta t}{2} \epsilon \psi_i^{(1)}, \quad (\text{A.8})$$

1080 where the corrective term ψ_i has been expanded to the first order. In addition, it can also be shown that:

$$g_i^{neq} = g_i - g_i^{eq} = \left(1 + \frac{\Delta t}{2\tau} \right) \left(\sum_{n=0}^{\infty} \epsilon^n f_i^{(n)} - f_i^{eq} \right) = \frac{\bar{\tau}}{\tau} \left(\sum_{n=0}^{\infty} \epsilon^n f_i^{(n)} - f_i^{eq} \right). \quad (\text{A.9})$$

1081 Substituting Equations (A.6), (A.8) and (A.9) into Equation (A.3), the resulting equation can be sep-
 1082 arated according to the different orders of ϵ leading to an infinite set of cascaded equations. The target
 1083 equations being here the Navier-Stokes equations, a first order Chapman-Enskog expansion in ϵ is sufficient
 1084 [74]. Therefore, by keeping only the first two equations (i.e. the ones corresponding to $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon)$), the
 1085 following relations are deduced:

$$0 = -\frac{1}{\tau} \left(f_i^{(0)} - f_i^{eq} \right) + \mathcal{O}(\Delta t^2), \quad (\text{A.10})$$

1086 and,

$$\left(\left[\frac{\partial}{\partial t_1} + \boldsymbol{\xi}_i \frac{\partial}{\partial \mathbf{x}_1} \right] f_i^{(0)} - \psi_i \right) = -\frac{1}{\tau} f_i^{(1)} + \mathcal{O}(\Delta t^2). \quad (\text{A.11})$$

1087 In the following, the first-order approximation of the off-equilibrium part of the distribution functions given
 1088 by Equation (A.11) will be denoted by $f_i^{neq,(1)}$.

1089 Appendix B. Reconstructed off-equilibrium contribution in the case of the HRR-LBM

1090 Malaspinas derived in [49] an elegant recursive relation for the off-equilibrium moments in the context of
 1091 the athermal Boltzmann-BGK equation. In this Appendix, it is proposed to use the same formalism in order

1092 to show that the reconstructed off-equilibrium component in the hybrid lattice Boltzmann - Navier-Stokes
 1093 method is genuinely contained in the HRR lattice Boltzmann scheme.

1094 As introduced in Section 2.2.2, both the equilibrium and the off-equilibrium distribution functions of the
 1095 bulk lattice Boltzmann solver are expanded using the Hermite formalism. Therefore, to ensure consistency,
 1096 one can expand f_i^{eq} and $f_i^{neq,(1)}$ appearing in Equation (21) in the same way, such as:

$$f_i^{eq} = w_i \sum_{n=0}^N \frac{1}{c_s^{2n} n!} \mathbf{a}_0^{(n)} : \mathcal{H}_i^{(n)} \quad \text{and} \quad f_i^{neq,(1)} = w_i \sum_{n=0}^{N_r} \frac{1}{c_s^{2n} n!} \mathbf{a}_1^{(n)} : \mathcal{H}_i^{(n)}, \quad (\text{B.1})$$

1097 In Equation (B.1), $\mathbf{a}_0^{(n)}$ and $\mathbf{a}_1^{(n)}$ are the equilibrium and off-equilibrium Hermite coefficients defined as:

$$\mathbf{a}_0^{(n)} = \sum_{i=1}^q f_i^{eq} \mathcal{H}_i^{(n)} \quad \text{and} \quad \mathbf{a}_1^{(n)} = \sum_{i=1}^q f_i^{neq,(1)} \mathcal{H}_i^{(n)}. \quad (\text{B.2})$$

1098 By projecting Equation (21) onto the basis of Hermite polynomials and using Rodrigues' formula [75], it
 1099 simplifies into:

$$\frac{\partial}{\partial t} a_{0,\alpha_1 \dots \alpha_n}^{(n)} + \frac{\partial}{\partial \alpha_{n+1}} a_{0,\alpha_1 \dots \alpha_n \alpha_{n+1}}^{(n+1)} + c_s^2 \sum_{i=1}^n \frac{\partial}{\partial \alpha_i} a_{0,\bar{\alpha}_i}^{(n-1)} + \delta_{2n} a_{\psi,\alpha_1 \alpha_2}^{(2)} = -\frac{1}{\tau} a_{1,\alpha_1 \dots \alpha_n}^{(n)}, \quad (\text{B.3})$$

1100 where the notation $\bar{\alpha}_i = \alpha_1 \dots \alpha_{i-1} \alpha_{i+1} \dots \alpha_n$ has been introduced for the sake of clarity. Equation (B.3) is
 1101 almost the same as the one derived in [49] except it exhibits a contribution due to the corrective term ψ_i
 1102 through $a_{\psi,\alpha_1 \alpha_2}^{(2)}$. The most interesting point about this term is that it only contributes to Equation (B.3)
 1103 when $n = 2$. Indeed, as shown by Equation (12), the corrective term is defined with 2nd-order Hermite
 1104 polynomials. Yet, by virtue of the orthogonality properties of Hermite polynomials [75], one directly has
 1105 $a_{\psi,\alpha_1 \dots \alpha_n}^{(n)} = 0$ if $n \neq 2$ which justifies the use of the Kronecker delta δ_{2n} as a prefactor.

1106 After some algebra (we refer the reader to [49] for the calculation steps), this relation can be further
 1107 simplified as:

$$a_{1,\alpha_1 \dots \alpha_n}^{(n)} = u_{\alpha_n} a_{1,\alpha_1 \dots \alpha_{n-1}}^{(n-1)} + \sum_{i=1}^{n-1} u_{\alpha_1 \dots \alpha_{\alpha_{n-2}}} a_{1,\alpha_i \alpha_n}^{(2)} \quad \text{for } n \geq 3. \quad (\text{B.4})$$

1108 Again, as for Equation (B.3), the resulting recursive relation is almost the same as the one derived in [49].
 1109 However, this time, the difference is implicit as it occurs in the definition of $\mathbf{a}_1^{(2)}$ which constitutes the
 1110 initialisation step of the recursive formula. Indeed, in [49] the Hermite coefficient $\mathbf{a}_1^{(2)}$ was computed thanks
 1111 to a Chapman-Enskog expansion leading to $a_1^{(2)} = -2\rho\tau c_s^2 \mathbf{S} + \mathcal{O}(\text{Ma}^3)$ where $\mathbf{S} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. The cubic
 1112 Mach error term (which does not appear in the Navier-Stokes equations) was then neglected thanks to a low
 1113 Mach number approximation. However, in the present case, the ψ_i corrective term is specifically designed
 1114 such as to remove the cubic Mach error term. Therefore, after a Chapman-Enskog expansion one exactly
 1115 obtains $a_1^{(2)} = -2\rho\tau c_s^2 \mathbf{S}$ which means that all the quadrature-related error terms are properly discarded and
 1116 that no approximation regarding the Mach number has to be made.

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