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Zonotope-based Fault Estimation for Uncertain Discrete-Time Switched Linear Systems

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Abstract: This paper investigates actuator fault estimation for discrete-time switched linear systems subject to unknown but bounded disturbances and measurement noises. By taking the fault as an auxiliary state vector, the proposed method consists of two steps. First, a switched L_∞-based descriptor observer attenuating the effect of uncertainties is designed to obtain simultaneous point estimate of the system state and the fault term. Second, interval estimation is achieved by integrating robust point estimation with zonotopic analysis techniques. The observer gains are calculated by solving Linear Matrix Inequality (LMI) derived using a common Lyapunov function. Finally, the effectiveness of the proposed approach is demonstrated through a numerical example.

Keywords: Fault estimation, discrete-time switched linear systems, zonotope, peak-to-peak analysis, LMIs.

1. INTRODUCTION

Fault estimation, as a significant phase of fault diagnosis, aims to determine further information about fault occurred in dynamic systems such as duration, magnitude and location (Zhang et al. (2013)) and to avoid human operator and material damage. Several approaches of fault estimation have been developped for some families of dynamical systems in the literature (see e.g., Jiang et al. (2011), Ze-Hui and Jiang (2007), Witczak et al. (2016), and the references therein). Switched systems are one class of hybrid dynamical systems (Liberzon and Morse (1999)), consisting of a family of subsystems and a switching rule that operates switching between them. The motivation of studying switched systems can be interpreted from the fact that they can describe a lot of complex real-life phenomena. The problem of fault estimation and fault-tolerant control for switched systems has attracted the interest of many researchers (Du and Jiang (2016); Du et al. (2017); Han et al. (2019).

In parallel, interval fault estimation for switched systems has gained a considerable attention. Interval techniques compute the set of all admissible values and provide two bounds which enclose the real state (Ethabet et al. (2018)). In Zhang et al. (2019), an actuator fault-based interval estimation method for discrete-time linear systems is proposed. In Garbouj et al. (2020) for switched Takagi-Sugeno systems, an interval observer-based fault estimation method is introduced. Under the assumption that disturbances are unknown but bounded, a simultaneous interval state and fault estimation for continuous-time switched systems is developed in Zammali et al. (2022). Marouani et al. (2021) proposes an H_∞ interval observer for discrete-time switched systems to jointly estimate the state and the unknown input.

On the other hand, many research studies have been drawn to zonotope-based approaches in recent years because of their flexibility and efficiency. In Zhou et al. (2019), a zonotope-based interval fault estimation method for discrete-time LPV descriptor systems is proposed. Using a state augmentation approach, a zonotope-based fault estimation approach is investigated in Zhang et al. (2020b) for descriptor systems. Motivated by those work, the main goal of this paper is to extend zonotope-based estimation in order to estimate actuator faults for discrete-time switched linear systems subject to uncertainties. By defining an augmented state including the state vector and the fault term, the proposed interval estimation approach combines the so-called TNL observer design (where T, N and L denote the weighting matrices and gain used in this strategy) with zonotopic analysis. To clarify, first, a robust observer based on peak-to-peak analysis is developed for augmented state point estimation. The observer gain computation is expressed in terms of LMIs using L_∞ criterion. Following that, zonotopic analysis is used to help obtain the interval for the above-mentioned augmented state.

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The rest of this paper is organized as follows. Some preliminaries results including properties and definitions are introduced in Section 2. In Section 3, the problem statement is provided. Main results are described in Section 4. In Section 5, simulation results are shown to illustrate the efficiency of the proposed approach. Section 6 concludes the paper and gives some perspectives.

2. PRELIMINARIES

The following standard notations and definitions are used in this paper. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the sets of natural numbers, integers and real numbers. The set of positive real numbers and positive integers are denoted by $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ respectively. $|x|$ denotes the elementwise absolute value of a vector $x \in \mathbb{R}^n$. I denotes the identity matrix with any dimension. 0 represents a zero number, vector or matrix with appropriate dimension. $\underline{x}$ and $\overline{x}$ are the lower and upper bounds of a variable x such that $\underline{x} \leq x \leq \overline{x}$. The matrices A^T and A^+, with $A \in \mathbb{R}^{n\times m}$, denote respectively the transpose and the Moore-Penrose pseudo-inverse of matrix A. The relation $P \prec 0$ ($P \succ 0$) indicates that $P = P^T$ is strictly negative (positive) definite. The comparison operators $\leq$ and $\geq$ should be understood elementwise for vectors and matrices. The operators $\oplus$ and $\odot$ represent the Minkowski sum and the linear image operators, respectively. The asterisk $*$ denotes the symmetric term in a symmetric matrix. The following properties and definitions are used in this paper.

Definition 1. (Combastel (2003)) An s-order zonotope $\mathrm{Z} \subset \mathbb{R}^n$ is an affine transformation of a hypercube $\mathrm{B}^s = [-1,\ 1]^s$ in $\mathbb{R}^n$ which can be expressed as follows:

$$\mathrm{Z} = \langle p, H\rangle = p + H\mathrm{B}^s = \{z \in \mathbb{R}^n : z = p + Hb\} \quad (1)$$

where $p \in \mathbb{R}^n$ denotes the center of Z and $H \in \mathbb{R}^{n\times s}$ is called the generator matrix of Z.

Definition 2. (Zhang et al. (2020a)) For a set $\mathrm{Z} \subset \mathbb{R}^n$, its interval hull $\mathrm{Box}(Z)$ is defined as the smallest interval vector containing it, which is denoted as follows:

$$\mathrm{Z} \subseteq \mathrm{Box}(\mathrm{Z}) = [\underline{z}, \overline{z}] \quad (2)$$

where $[\underline{z}, \overline{z}] = \{z \in \mathrm{Z},\ \underline{z} < z < \overline{z}\}$ is an interval vector, $\underline{z}$ and $\overline{z}$ denote the lower and upper bounds of z.

Property 1. (Combastel (2003)) Consider two sets $\mathrm{Z}_1 = \langle p_1, H_1\rangle$ and $\mathrm{Z}_2 = \langle p_2, H_2\rangle$. The Minkowski sum of Z_1 and Z_2 is defined as:

$$\mathrm{Z}_1 \oplus \mathrm{Z}_2 = \langle p_1 + p_2\,,\, [H_1 \quad H_2]\rangle\,, \quad (3)$$

where $p_1, p_2 \in \mathbb{R}^n$ are known vectors and $H_1, H_2 \in \mathbb{R}^{n\times s}$ are determined matrices.

Property 2. (Combastel (2003)) The linear product of a zonotope $\mathrm{Z} = \langle p, H\rangle$ and a matrix $K \in \mathbb{R}^{m\times n}$ is denoted as $\odot$ and defined as:

$$K \odot \mathrm{Z} = \langle Kp\,,\, KH\rangle\,. \quad (4)$$

Property 3. (Zhang et al. (2020a)) For a zonotope $\mathrm{Z} = \langle p, H\rangle \subset \mathbb{R}^n$, its interval hull $\mathrm{Box}(\mathrm{Z}) = [\underline{z}, \overline{z}]$ can be obtained by

$$\begin{cases} \underline{z}_i = p_i - \sum_{j=1}^{s} |H_{i,j}|\,, & i = 1, ..., n \\ \overline{z}_i = p_i + \sum_{j=1}^{s} |H_{i,j}|\,, & i = 1, ..., n \end{cases}, \quad (5)$$

with s is the column number of H. According to Definitions 1 and 2, the interval hull of $\mathrm{Z} = \langle p, H\rangle$ can also be computed by

$$\mathrm{Z} = p \oplus H\mathrm{B}^s \subseteq \mathrm{Box}(Z) = p \oplus \overline{H}\mathrm{B}^n, \quad (6)$$

where $\overline{H} \in \mathbb{R}^{n\times n}$ is a diagonal matrix as follows:

$$\overline{H} = \mathrm{Diag}\left(\left[\sum_{j=1}^{s} |H_{1,j}| ... \sum_{j=1}^{s} |H_{n,j}|\right]\right).$$

In the application of zonotopic techniques, the order of the zonotope will linearly increase. Due to this, a useful reduction method has been proposed in Combastel (2003) in order to control the order of the zonotope. The following property is introduced to describe this reduction operator.

Property 4. (Combastel (2003)) A high-dimensional zonotope $\mathrm{Z} \subset \mathbb{R}^n$, can be bounded by a lower dimensional one via a reduction operator

$$\mathrm{Z} = \langle p, H\rangle \subseteq \langle p, \downarrow_l (H)\rangle \subseteq \mathrm{Box}(\mathrm{Z})\,, \quad n < l < s \quad (7)$$

where $p \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n\times s}$ are the center and the generation matrix of Z. l is the maximum number of columns of the generated matrix after reduction. $\downarrow_l (H)$ denotes the complexity reduction operator with $n < l < s$.

Lemma 1. (Wang et al. (2015)) Given three matrices $A \in \mathbb{R}^{a\times b}$, $B \in \mathbb{R}^{b\times c}$ and $C \in \mathbb{R}^{a\times c}$. If $\mathrm{rank}(B) = c$, then the general solution of the equation $AB = C$ is defined by:

$$A = CB^+ + S(I - BB^+) \quad (8)$$

where $S \in \mathbb{R}^{a\times b}$ is an arbitrary matrix.

Lemma 2. (Schur complement)
Given the matrices $R = R^T$, $Q = Q^T$ and S with appropriate dimensions. The following LMIs are equivalent:

i) $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0.$

ii) $R \succ 0; Q - SR^{-1}S^T \succ 0.$

iii) $Q \succ 0; R - S^T Q^{-1} S \succ 0.$

3. PROBLEM STATEMENT

Consider the following discrete-time switched linear system

$$\begin{cases} x_{k+1} = A_q x_k + B_q u_k + F_q f_k + w_k \\ y_k = Cx_k + v_k \end{cases}, q \in \overline{1, N} \quad (9)$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $y \in \mathbb{R}^{n_y}$ is the output and $u \in \mathbb{R}^{n_u}$ is the control input. $w \in \mathbb{R}^{n_w}$ and $v \in \mathbb{R}^{n_v}$ are the disturbances and the measurement noise assumed to be bounded, $f \in \mathbb{R}^{n_f}$ is an additive fault. We denote respectively by A_q, B_q and C the state matrices, the input matrices and the output matrices with appropriate dimensions. Besides, F_q denote the fault distribution matrices describing the directions of the actuator fault vector. The index q specifies, at each instant, the system that is currently followed and N is the number of linear subsystems. Let $\{t_q\}$ denote the switching time instants. In this paper, it is supposed that the active subsystem and the switching instants are known. The state x and the fault f are assumed to be unknown.

The following assumption is considered.

Assumption 1. The initial state x_0, the disturbances w_k and the measurement noise v_k are supposed to be unknown but bounded by known zonotopes:

$$x_0 \in \chi_0 = \langle \, p_0 \, , \; H_0 \, \rangle$$
$$w_k \in \mathcal{W} = \langle \, 0 \, , \; H_w \, \rangle \qquad (10)$$
$$v_k \in \mathcal{V} = \langle \, 0 \, , \; H_v \, \rangle$$

where p_0 is a known vector, H_0, H_w and H_v are known diagonal matrices.

The goal of this work is to compute an interval vector $\left[\underline{f}_k, \overline{f}_k\right]$ containing the actuator fault f_k such that

$$\underline{f}_k \leq f_k \leq \overline{f}_k \,, \quad \forall k \in \mathbb{Z}_+.$$

In this paper, the main goal is to design a zonotope-based fault and state interval estimation for the discrete-time switched linear system (9). For this purpose, an augmented descriptor system is constructed. It is obtained by considering the fault as an auxiliary state. Then, for this augmented system, a L_∞ switched descriptor observer is proposed to jointly estimate the original state x_k and the fault f_k.

In the following, we define the augmented state vector as:

$$\tilde{x}_{k+1} = \begin{bmatrix} x_{k+1} \\ f_k \end{bmatrix} \,, \qquad (11)$$

then, the system (9) can be rewritten as a switched descriptor system

$$\begin{cases} E_q \tilde{x}_{k+1} = \tilde{A}_q \tilde{x}_k + \tilde{B}_q u_k + \tilde{I} w_k, \\ \qquad y_k = \tilde{C} \tilde{x}_k + v_k, \end{cases} \qquad (12)$$

where the matrices E_q, $\tilde{A}_q$, $\tilde{B}_q$, $\tilde{I}$ and $\tilde{C}$ are given by

$$E_q = \begin{bmatrix} I & -F_q \\ 0 & 0 \end{bmatrix}, \; \tilde{A}_q = \begin{bmatrix} A_q & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{B}_q = \begin{bmatrix} B_q \\ 0 \end{bmatrix},$$
$$\tilde{I} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; \tilde{C} = [\, C \;\; 0 \,] \,.$$

Note that, the state $\tilde{x}$ of (12) consists of the state vector x_k and the actuator fault f_k. Hence, the interval estimation problem for (9) is replaced by the interval estimation design for (12).

4. MAIN RESULTS

In this section, a new method is investigated to design interval fault estimation based on TNL structure and zonotopic analysis. We propose the following observer for (12)

$$\begin{cases} \xi_{k+1} = T_q \tilde{A}_q \hat{\tilde{x}}_k + T_q \tilde{B}_q u_k + L_q (y_k - \tilde{C} \hat{\tilde{x}}_k), \\ \hat{\tilde{x}}_k = \xi_k + N_q y_k, \end{cases} \qquad (13)$$

where ξ_k is an intermediate variable and $\hat{\tilde{x}}_k$ denotes the augmented state estimation. The matrices $L_q \in \mathbb{R}^{(n_x+n_f)\times n_y}$ are the observer gains to be computed later. The weighted matrices T_q and N_q are constant matrices to be computed satisfying the following condition:

$$T_q E_q + N_q \tilde{C} = I. \qquad (14)$$

The relation (14) can be rewritten as follows

$$[\, T_q \;\; N_q \,] \begin{bmatrix} E_q \\ \tilde{C} \end{bmatrix} = I. \qquad (15)$$

Based on Lemma 1, the general solution of (14) is given by

$$[\, T_q \;\; N_q \,] = \begin{bmatrix} E_q \\ \tilde{C} \end{bmatrix}^+ + S_q \left(I - \begin{bmatrix} E_q \\ \tilde{C} \end{bmatrix} \begin{bmatrix} E_q \\ \tilde{C} \end{bmatrix}^+ \right). \qquad (16)$$

Remark 1. The major difference between the observer designed in (13) and the point estimator in Zhang et al. (2020a) is the presence of additional matrices T_q and N_q. Note that if we choose $T_q = I$ and $N_q = 0$, observer (13) reduces to the observer proposed in Zhang et al. (2020a). In fact, by introducing these matrices, the proposed method provides more design degrees of freedom.

To design and analyze the observer (13), we define:

$$e_k = \tilde{x}_k - \hat{\tilde{x}}_k. \qquad (17)$$

Using equation (14), from (12), the descriptor vector $\tilde{x}_k$ satisfies

$$\tilde{x}_{k+1} = T_q \tilde{A}_q \tilde{x}_k + T_q \tilde{B}_q u_k + T_q \tilde{I} w_k + N_q \tilde{C} \tilde{x}_{k+1} \qquad (18)$$

It follows that the error dynamics are then

$$e_{k+1} = (T_q \tilde{A}_q - L_q \tilde{C}) e_k + T_q \tilde{I} w_k - L_q v_k - N_q v_{k+1}. \quad (19)$$

To facilitate the observer design, (19) is rewritten as

$$e_{k+1} = \mathcal{A}_q e_k + \mathcal{D}_q d_k, \qquad (20)$$

with

$$\mathcal{A}_q = T_q \tilde{A}_q - L_q \tilde{C}, \;\; \mathcal{D}_q = [\, I \;\; L_q \;\; N_q \,] \,, \qquad (21)$$

$$d_k = \begin{bmatrix} T_q \tilde{I} w_k \\ -v_k \\ -v_{k+1} \end{bmatrix}. \qquad (22)$$

4.1 Robust descriptor observer design based L_∞ approach

This part proposes a robust observer design for the augmented switched descriptor system defined by (12). Stability analysis is performed by employing a common Lyapunov function under an arbitrary switching signal. The proposed design is robust against uncertainties. We propose the following Theorem in order to determine the weighting matrices T_q, N_q and L_q such that the estimation error in (20) is stable and satisfies the L_∞ performance.

Theorem 1. Given a scalar $0 < \lambda < 1$, the error dynamics system in (20) is stable and satisfies the following L_∞ performances

$$\|e_k\| \leq \sqrt{\gamma \left(\lambda (1-\lambda)^k V(e_0) + \gamma \, \|d\|_\infty^2 \right)} \qquad (23)$$

if there exist constants $\gamma > 0$, $\rho > 0$, a matrix $P = P^T > 0$, $P \in \mathbb{R}^{(n_x+n_f)\times(n_x+n_f)}$ and constant matrices $W_q \in \mathbb{R}^{(n_x+n_f)\times n_y}$ and $Y_q \in \mathbb{R}^{(n_x+n_f)\times(n_x+n_f+n_y)}$ such that for all $q \in \overline{1, N}$,

$$\begin{bmatrix} (\lambda-1)P & * & * & * & * \\ 0 & -\rho I & * & * & * \\ 0 & 0 & -\rho I & * & * \\ 0 & 0 & 0 & -\rho I & * \\ \Omega_{1q} & P & W_q & \Omega_{2q} & -P \end{bmatrix} \prec 0 \qquad (24)$$

$$\begin{bmatrix} \lambda P & * & * \\ 0 & (\gamma-\rho) I & * \\ I & 0 & \gamma I \end{bmatrix} \succ 0 \qquad (25)$$

where

$\Omega_{1q} = P\Theta_q^+\alpha_1\tilde{A}_q + Y_q\psi_q\alpha_1\tilde{A}_q - W_q\tilde{C}^T$

$\Omega_{2q} = P\Theta_q^+\alpha_2 + Y_q\psi_q\alpha_2$

$\alpha_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\Theta_q = \begin{bmatrix} E_q \\ \tilde{C} \end{bmatrix}$, $\psi_q = I - \Theta_q\Theta_q^+$.

The observer gains L_q, the matrices T_q and N_q are given by

$$\begin{cases} L_q = P^{-1}W_q \\ T_q = \Theta_q^+\alpha_1 + P^{-1}Y_q\psi_q\alpha_1 \\ N_q = \Theta_q^+\alpha_2 + P^{-1}Y_q\psi_q\alpha_2 \end{cases} . \tag{26}$$

Proof. For the system (20), consider the Lyapunov function $V_k = e_k^T P\, e_k$. The increment of V_k is

$$V_{k+1} - V_k = \begin{bmatrix} e_k \\ d_k \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_q^T P\mathcal{A}_q - P & \mathcal{A}_q^T P\mathcal{D}_q \\ * & \mathcal{D}_q^T P\mathcal{D}_q \end{bmatrix} \begin{bmatrix} e_k \\ d_k \end{bmatrix} \tag{27}$$

Bearing in mind that $P\mathcal{D}_q = [\, P \; W_q \; \Omega_{2q} \,]$ and $\Omega_{1q} = P\left[\left(\Theta_q^+\alpha_1 + P^{-1}Y_q\psi_q\alpha_1\right)\tilde{A}_q - P^{-1}W_q\tilde{C}^T\right] = P\mathcal{A}_q$, the LMI (24) is equivalent to

$$\begin{bmatrix} (\lambda-1)P & * & * \\ 0 & -\rho I & * \\ P\mathcal{A}_q & P\mathcal{D}_q & -P \end{bmatrix} \prec 0 \tag{28}$$

Now, consider the following matrix

$$\begin{bmatrix} I & 0 & \mathcal{A}_q^T \\ 0 & I & \mathcal{D}_q^T \end{bmatrix} \tag{29}$$

Pre- and post- multiplying (28) with (29), we obtain:

$$\begin{bmatrix} \mathcal{A}_q^T P\mathcal{A}_q - P & \mathcal{A}_q^T P\mathcal{D}_q \\ * & \mathcal{D}_q^T P\mathcal{D}_q \end{bmatrix} + \begin{bmatrix} \lambda P & 0 \\ 0 & -\rho I \end{bmatrix} \prec 0 \tag{30}$$

Pre- and post- multiplying (30) with $\left[\, e_k^T \; d_k^T \,\right]$ and its transpose, it follows that:

$$\begin{bmatrix} e_k \\ d_k \end{bmatrix}^T \begin{bmatrix} \mathcal{A}_q^T P\mathcal{A}_q - P & \mathcal{A}_q^T P\mathcal{D}_q \\ * & \mathcal{D}_q^T P\mathcal{D}_q \end{bmatrix} \begin{bmatrix} e_k \\ d_k \end{bmatrix} \tag{31}$$

$$+ \begin{bmatrix} e_k \\ d_k \end{bmatrix}^T \begin{bmatrix} \lambda P & 0 \\ 0 & -\rho I \end{bmatrix} \begin{bmatrix} e_k \\ d_k \end{bmatrix} \prec 0 \tag{32}$$

Therefore, we obtain

$$V_{k+1} - V_k + \lambda e_k^T P e_k - \rho d_k^T d_k < 0 \tag{33}$$

Then, the inequality (33) can be rewritten as

$$V_{k+1} - V_k < -\lambda V_k + \rho d_k^T d_k \tag{34}$$

when $w_k = 0$ and $v_k = 0$, then $d_k = 0$ and (34) is equivalent to

$$V_{k+1} - V_k < -\lambda V_k < 0 \tag{35}$$

Consequently, the error system in (20) is stable when no uncertainties. Moreover, inequality (34) is equivalent to

$$V_{k+1} < (1-\lambda)V_k + \rho\, \|d\|_\infty^2 \tag{36}$$

From (36) , one can obtain

$$\begin{aligned} V_k &\le (1-\lambda)^k V_0 + \rho \sum_{\tau=0}^{k-1} (1-\lambda)^\tau \|d\|_\infty^2 \\ &\le (1-\lambda)^k V_0 + \rho\frac{(1-\lambda^k)}{\lambda} \|d\|_\infty^2 \\ &\le (1-\lambda)^k V_0 + \frac{\rho}{\lambda} \|d\|_\infty^2 \end{aligned} \tag{37}$$

Based on the Schur complement in Lemma 2, the inequality (25) is equivalent to

$$\begin{bmatrix} \lambda P & * \\ 0 & (\gamma-\rho)I \end{bmatrix} - \frac{1}{\gamma}\begin{bmatrix} I \\ 0 \end{bmatrix} [\, I \; 0 \,] \succ 0 \tag{38}$$

After, pre- and post- multiplying (38) with $\left[\, e_k^T \; d_k^T \,\right]$ and its transpose, we get

$$e_k^T e_k \le \gamma\left(\lambda V_k + (\gamma-\rho)\, \|d\|_\infty^2\right) \tag{39}$$

According to (37), inequality in (39) gives

$$\begin{aligned} e_k^T e_k &\le \gamma\left(\lambda\left((1-\lambda)^k V_0 + \frac{\rho}{\lambda}\|d\|_\infty^2\right) + (\gamma-\rho)\, \|d\|_\infty^2\right) \\ &\le \gamma\left(\lambda(1-\lambda)^k V_0 + \gamma\, \|d\|_\infty^2\right) \end{aligned} \tag{40}$$

which follows $\|e\| \le \sqrt{\gamma\left(\lambda(1-\lambda)^k V_0 + \gamma\, \|d\|_\infty^2\right)}$. Therefore, the L_∞ criterion (23) is satisfied.

4.2 State and fault interval estimation

The goal of this section is to achieve interval estimation for (12) using zonotopic techniques based on the designed observer (13). We begin by computing an approximation set of the error e_k given in (19) using known zonotopic bounds of disturbances and measurement noise presented in Assumption 1, and then we deduce the bounds of the augmented state vector (11). From (17), we have

$$\tilde{x} = \hat{\tilde{x}}_k + e_k. \tag{41}$$

If one can find $\overline{e}_k$ and $\underline{e}_k$ such that $\underline{e}_k \le e_k \le \overline{e}_k$, then from (41), we can reach our goal by deriving the interval estimation of $\tilde{x}$ as follows

$$\begin{cases} \overline{\tilde{x}}_k = \hat{\tilde{x}}_k + \overline{e}_k \\ \underline{\tilde{x}}_k = \hat{\tilde{x}}_k + \underline{e}_k \end{cases} \tag{42}$$

where $\overline{\tilde{x}}$ and $\underline{\tilde{x}}$ are respectively the upper and the lower bounds of the augmented state $\tilde{x}$.

Since the initial state satisfies $x_0 \in \mathcal{X}_0 = \langle\, p_0,\, H_0 \,\rangle$, considering $\hat{\tilde{x}}_0 = p_0$ deduces that the initial error $e_0 \in \mathcal{X}_0 \oplus (-\hat{\tilde{x}}_0) = \langle\, 0,\, H_0 \,\rangle \triangleq \mathcal{Z}_0^e$. We introduce the following theorem in order to compute a zonotopic set $e_{k+1} \in Z_{k+1} = \langle\, 0,\, H_{k+1}^e \,\rangle$.

Theorem 2. Assume that $e_k \in Z_k^e = \langle\, 0 \; H_k^e \,\rangle$. If Assumption 1 is satisfied, then there exists a zonotope Z_{k+1}^e such that $e_{k+1} \in Z_{k+1}^e = \langle\, 0 \; H_{k+1}^e \,\rangle$ where

$$H_{k+1}^e = \left[\, (T_q\tilde{A}_q - L_q\tilde{C})\downarrow_q(H_k^e) \;\; T_q\tilde{I}H_w \;\; -L_qH_v \;\; -N_qH_v \,\right] \tag{43}$$

Proof. Since at the step k, $e_k \in Z_k^e$, the disturbances and the measurement noise satisfy $w_k \in \mathcal{W}$, $v_k \in \mathcal{V}$ and $v_{k+1} \in \mathcal{V}$, (19) gives

$$Z_{k+1}^e = (T_q\tilde{A}_q - L_q\tilde{C})Z_k^e \oplus T_q\tilde{I}\mathcal{W} \oplus (-L_q)\mathcal{V} \oplus (-N_q)\mathcal{V}. \tag{44}$$

Thus,

$$\begin{aligned} \left\langle 0, H_{k+1}^e \right\rangle = (T_q\tilde{A}_q - L_q\tilde{C})\left\langle 0, H_k^e \right\rangle \oplus (T_q\tilde{I})\left\langle 0, H_w \right\rangle \\ \oplus(-L_q)\left\langle 0, H_v \right\rangle \oplus (-N_q)\left\langle 0, H_v \right\rangle. \end{aligned} \tag{45}$$

Using the properties of zonotopes, one can get

$$H_{k+1}^e = \left[\, (T_q\tilde{A}_q - L_q\tilde{C})(H_k^e) \;\; T_q\tilde{I}H_w \;\; -L_qH_v \;\; -N_qH_v \,\right] \tag{46}$$

As mentioned in Section 2, the column number of the matrix H_k^e linearly increases. This can lead to computational complexity. As a solution for this problem, we limit the column number to a fixed one by using the reduction operator $\downarrow_q(.)$. The use of this operator satisfies the inclusion property $\langle 0, H_k^e \rangle \subseteq \langle 0, \downarrow q(H_k^e) \rangle$. Consequently, by replacing the matrix H_k^e by the limit $\downarrow q(H_k^e)$ in (46), we obtain (43).

Now, using Property 3, the interval estimation of the augmented state can be calculated as follows

$$\begin{cases} \underline{\tilde{x}}_k(i) = \hat{\tilde{x}}_k(i) - \sum_{j=1}^{n_h} |H_k^e(i,j)| \; , i = 1...(n_x + n_f) \\ \overline{\tilde{x}}_k(i) = \hat{\tilde{x}}_k(i) + \sum_{j=1}^{n_h} |H_k^e(i,j)| \; , i = 1...(n_x + n_f) \end{cases}, \quad (47)$$

with n_h is the column number of H_k^e.

Since the state vector of the augmented descriptor system $\tilde{x}$ consists of the state x and the actuator fault f. Then from (11), we can simply calculate the two vectors x and f as $\begin{cases} x_k = \beta_1 \tilde{x}_k \\ f_{k-1} = \beta_2 \tilde{x}_k \end{cases}$, with $\beta_1 = [I \quad 0]$ and $\beta_2 = [0 \quad I]$. Finally, we deduce that

$$\begin{cases} \hat{x}_k = \beta_1 \hat{\tilde{x}}_k \\ \overline{x}_k = \beta_1 \overline{\tilde{x}}_k \\ \underline{x}_k = \beta_1 \underline{\tilde{x}}_k \end{cases}, \text{ and } \begin{cases} \hat{f}_{k-1} = \beta_2 \hat{\tilde{x}}_k \\ \overline{f}_{k-1} = \beta_2 \overline{\tilde{x}}_k \\ \underline{f}_{k-1} = \beta_2 \underline{\tilde{x}}_k \end{cases}.$$

where $\underline{x}_k$, $\overline{x}_k$, $\underline{f}_k$ and $\overline{f}_k$ are the lower and the upper bounds of the state and of the actuator fault, respectively.

5. A NUMERICAL EXAMPLE

In this section, the performances of the proposed approach are illustrated via a numerical example. We consider the discrete-time switched linear system (9) defined with three subsystems, i.e., $N = 3$, where

$$A_1 = \begin{bmatrix} 0.55 & 0.5 & 0.7 \\ 0 & 0.8 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.44 & -0.4 & -0.56 \\ 0 & -0.64 & -0.4 \\ 0 & 0 & -0.32 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.1 & 1 & 1 \\ 0 & 0.2 & -0.5 \\ 0 & 0 & -0.6 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0.8 \\ 0.5 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.4 \\ 0.6 \\ 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 1.8 \\ 2 \\ 2.3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1.5 \\ 2 \\ 2.3 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1.9 \\ 2 \\ 2.3 \end{bmatrix}.$$

In this example, w_k and v_k are uniformly distributed bounded signals such that $H_w = 0.08$ and $H_v = 0.1$. According to Assumption 1, we select $p_0 = [0 \; 0 \; 0 \; 0]^T$ and $H_0 = 0.1I$. Besides, to limit the column number of the generator matrix, we set the reduction order of $\downarrow_q (H_k^e)$ as $q = 10$. By solving the LMIs in Theorem 1, the matrices P, L_q, T_q and N_q can be computed as follows

$$P = \begin{bmatrix} 0.4736 & 0 & 0 & 0 \\ 0 & 0.4736 & 0 & 0 \\ 0 & 0 & 0.4736 & 0 \\ 0 & 0 & 0 & 0.4736 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.3198 & 0.2485 \\ 0.1826 & -0.0506 \\ -0.4083 & -0.2240 \\ 0.0270 & -0.2401 \end{bmatrix}, \; L_2 = \begin{bmatrix} -0.2313 & -0.2398 \\ -0.1908 & 0.0662 \\ 0.3310 & 0.2053 \\ 0.0198 & 0.2011 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0.2525 & 0.0487 \\ -0.0236 & -0.0324 \\ -0.2406 & -0.0327 \\ -0.2054 & -0.0214 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.5216 & 0.0808 & -0.4784 & 0.0001 \\ -0.1631 & 0.3344 & -0.1631 & 0.0000 \\ -0.4425 & -0.2428 & 0.5575 & 0.0001 \\ -0.3750 & 0.2687 & -0.3750 & -0.0001 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0.6136 & -0.0158 & -0.3864 & 0.0001 \\ -0.2382 & 0.4526 & -0.2382 & 0.0000 \\ -0.4892 & -0.2206 & 0.5108 & 0.0001 \\ -0.3675 & 0.1982 & -0.3675 & -0.0000 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0.0471 & 1.0511 & -0.9529 & 0.0000 \\ -0.0426 & 0.0895 & -0.0426 & -0.0001 \\ -0.0265 & -1.0944 & 0.9735 & -0.0001 \\ -0.1893 & -0.1024 & -0.1893 & 0.0000 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -0.0808 & 0.4784 \\ 0.6656 & 0.1631 \\ 0.2428 & 0.4425 \\ -0.2687 & 0.3750 \end{bmatrix}, \; N_2 = \begin{bmatrix} 0.0158 & 0.3864 \\ 0.5474 & 0.2382 \\ 0.2206 & 0.4892 \\ -0.1982 & 0.3675 \end{bmatrix},$$

$$\mathrm{N}_3 = \begin{bmatrix} -1.0511 & 0.9529 \\ 0.9105 & 0.0426 \\ 1.0944 & 0.0265 \\ 0.1024 & 0.1893 \end{bmatrix}.$$

Noted that the condition (14) is satisfied. In the simulation, we consider an actuator time varying fault scenario as

$$f_k = \begin{cases} 0.1 \sin(k) & \text{if } 15 \leq k \leq 50 \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

Under the switching signal in Fig. 3, simulation results are plotted in Fig. 1 and 2 where the dashed lines correspond to the lower and upper bounds and the solid lines present the real variables. Fig. 1 shows the estimated results of the actuator fault scenario (48) and Fig. 2 depicts the real state components and their estimated bounds under the actuator fault (48). The results all show the feasibility and effectiveness of our approach in fault and state interval estimation.

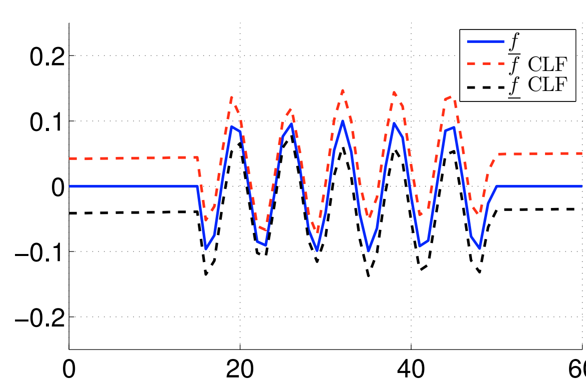

Fig. 1. Time-varying fault and its estimated bounds.

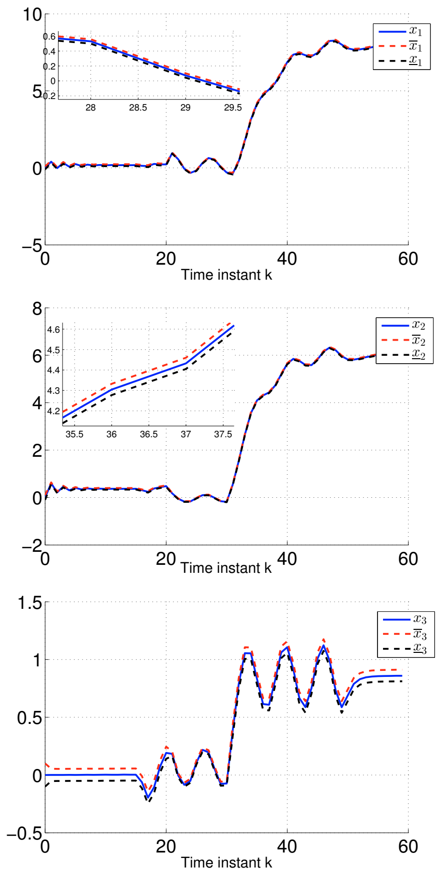

Fig. 2. State components and their estimated bounds.

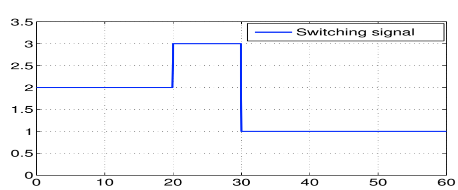

Fig. 3. Switching signal.

6. CONCLUSION

This paper presents a new approach to design zonotope-based jointly fault and state estimator for discrete-time switched linear systems subject to uncertainties. We combine a robust observer design with zonotopic analysis. Stability conditions of the proposed observer are ensured in terms of LMIs using a common Lyapunov function. Extending to multiple Lyapunov functions and considering unknown switching are perspectives for future work.

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